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Production Functions

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Isoelastic Elasticity of Substitution Production Functions*

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Abstract. We generalize the normalized Constant Elasticity of Substitution (CES) production function by allowing the elasticity of substitution to vary isoelastically with (i) relative factor shares, (ii) marginal rates of substitution, (iii) capital–labor ratios, or (iv) capital–output ratios. Ensuing four variants of Isoelastic Elasticity of Substitution (IEES) production functions have a range of intuitively desirable properties and yield empirically testable predictions for the functional relationship between relative factor shares and (raw or technology-adjusted) capital–labor ratios. As an empirical application, the parameters of IEES functions are estimated in a three-equation supply-side system with factor-augmenting technical change, based on data on aggregate production in the post-war US economy. Our estimates consistently imply that the elasticity of substitution between capital and labor has remained relatively stable, at about 0.8–0.9, from 1948 to the 1980s, followed by a period of secular decline.

Keywords: production function, factor share, elasticity of substitution, marginal rate of substitution, normalization.

JEL Classification Numbers: E23, E25, O33, O47.

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1 Introduction

The Constant Elasticity of Substitution (CES) production function, first introduced to economics by [Arrow, Chenery, Minhas, and Solow \(1961\)](#), is probably the most popular framework which allows factor shares to be affected by endogenous variables. The properties of an economy with CES production depend critically on the value of the elasticity of substitution σ . Whether the factors of production (say, capital and labor) are gross complements ($\sigma < 1$) or substitutes ($\sigma > 1$) is crucial both for long-run growth perspectives and short-run fluctuations of the economy. First, above-unity elasticity of substitution can be perceived as an engine of long-run endogenous growth ([Solow, 1956](#); [Jones and Manuelli, 1990](#); [Palivos and Karagiannis, 2010](#)). If capital and labor are gross substitutes then neither of them is essential for production, and thus physical capital accumulation alone can, under otherwise favorable circumstances, drive perpetual growth. Otherwise, the scarce factor limits the scope for economic development and output is bounded. Concurrently, the magnitude of the elasticity of substitution is also vital for the immediate impact of factor accumulation on factor shares. Under gross substitutes, accumulation of capital relative to labor increases the capital's share of output; under gross complements the opposite effect is observed. Hence, labor share declines observed across the world since the 1970-80s ([Karabarbounis and Neiman, 2014](#)) can be directly explained by capital deepening or capital-augmenting technological progress under CES production only if $\sigma > 1$.

The same caveat applies when dealing with other pairs of inputs. CES functions have been applied to issue of substitutability between exhaustible natural resources and accumulable physical capital ([Dasgupta and Heal, 1979](#); [Bretschger and Smulders, 2012](#)) or human capital (i.e., quality-adjusted labor, [Smulders and de Nooij, 2003](#)). No surprise that it is central to this literature whether these two inputs are gross complements ($\sigma < 1$) or substitutes ($\sigma > 1$), and thus if exhaustible resources are essential for production. CES functions have also been applied to the question of substitution possibilities between skilled and unskilled labor (e.g., [Caselli and Coleman, 2006](#)) as well as capital-skill complementarity ([Krusell, Ohanian, Ríos-Rull, and Violante, 2000](#); [Duffy, Papageorgiou, and Perez-Sebastian, 2004](#)). Whether σ is above or below unity determines whether capital accumulation and factor-augmenting technical change increase or depress the relative demand for skilled versus unskilled workers. The magnitude of σ is also important when discussing the substitutability among consumption goods in an agent's utility function, between intermediate goods in the production of a final good, or in the aggregation of domestic and imported goods by an open economy.

Empirical identification of σ is a notoriously difficult task, though. Looking at the estimates of the elasticity of substitution between capital and labor, we already observe disagreement. On the one hand, a voluminous literature exploiting time-series and

cross-firm variation for the USA (Antràs, 2004; Chirinko, 2008; Klump, McAdam, and Willman, 2007, 2012; Young, 2013; Oberfield and Raval, 2014) finds that the elasticity of substitution is below unity ($\sigma \approx 0.6 - 0.7$), and thus both factors of production are gross complements. On the other hand, numerous studies exploiting the cross-country variation in factor shares (Duffy and Papageorgiou, 2000; Piketty and Zucman, 2014; Piketty, 2014; Karabarbounis and Neiman, 2014) tend to imply gross substitutability, with $\sigma \approx 1.2 - 1.3$. Moreover, studies allowing for cross-country heterogeneity in σ find that it can be quite substantial (Duffy and Papageorgiou, 2000; Mallick, 2012).

But what if the key object at hand, the elasticity of substitution σ , is not constant after all? What if it depends on the capital–labor ratio k – either in raw or effective, technology-adjusted units – or on the capital–output ratio k/y ? Crucially, what if σ is systematically above unity for some configurations of factor endowments, and below unity for others?

Obviously, we are not the first to ask these questions. A substantial number of theoretical articles, proposing various production functions with variable elasticity of substitution, were published in the late 1960s and early 1970s. Next, after a three decade-long break, the topic re-emerged around 2000, with a much more empirical focus, fueled by the progress associated with production function normalization. Still, in our opinion, the literature has not managed so far to design a satisfactory framework for modeling endowment-specific elasticities of substitution. There are several loose ends hanging which we would like to pick up.

Our contribution to the literature is to put forward and thoroughly characterize a novel, tractable and empirically useful class of IsoElastic Elasticity of Substitution (IEES)¹ production functions. Our basic idea is simple. We design IEES functions so that they generalize the CES function in the same way as the CES function generalizes the Cobb–Douglas (Table 1): the Cobb–Douglas is isoelastic and implies constant factor shares, the CES function implies isoelastic factor shares and has a constant elasticity of substitution, whereas IEES functions have an isoelastic elasticity of substitution and a constant *elasticity of elasticity of substitution*. Moreover, just like both their predecessors, IEES functions are consistent with factor-augmenting technical change and exhibit globally constant returns to scale.

We consider four alternative variants of IEES functions by allowing the elasticity of substitution to vary isoelastically with (i) relative factor shares, (ii) marginal rates of substitution, (iii) capital–labor ratios, or (iv) capital–output ratios. Considering each of the four possibilities underscores that we remain agnostic in our choice of exact functional specification, at least in the space of two-input, constant-returns-to-scale production functions. It also signifies that the IEES class is quite versatile. Moreover,

¹Best pronounced as “yes”. Abbreviation designed to avoid confusion with the intertemporal elasticity of substitution (IES).

Table 1: How IEES Production Functions Generalize Cobb–Douglas and CES Functions

		Cobb-Douglas	CES	IEES
Output per worker	y	isoelastic		
Relative factor share	$\frac{\pi}{1-\pi}$	constant	isoelastic	
Elasticity of substitution	σ	1	constant	isoelastic
Elasticity of elasticity of substitution	ψ	0	0	constant

owing to the fact that all our calculations have been carried out in normalized units (de La Grandville, 1989; Klump and de La Grandville, 2000), not only is our basic idea simple, but also our analytical results remain sharp and are not cluttered with unnecessary algebra. Thanks to production function normalization the role of each parameter of IEES functions has been precisely disentangled from all others, facilitating theoretical discussions as well as parameter estimation (see Klump, McAdam, and Willman, 2012).

IEES production functions have a few notable advantages compared to functions with a variable elasticity of substitution (VES) which have already been analyzed in the literature. First, the class of IEES functions is sufficiently general to nest some of them directly, such as the Revankar’s VES (1971) or the Stone–Geary production function (Geary, 1949-50; Stone, 1954). In contrast to Revankar’s VES, most IEES functions allow σ to cross unity. This is crucial because it makes IEES functions useful in analyzing poverty traps and growth reversals: physical capital accumulation alone can become an engine of unbounded endogenous growth only if the elasticity of substitution $\sigma(k)$ exceeds unity, which in the IEES case may be true only for k sufficiently large. Second, as opposed to the empirically popular translog function (Christensen, Jorgenson, and Lau, 1973; Kim, 1992) or the empirically motivated VES function due to Lu (1967), it is not a local approximation of an arbitrary function but has well-behaved and economically interpretable properties globally. Third, as opposed to a recent idea to view the production function as an arbitrary spline of CES functions with different σ ’s (Antony, 2010), it implies that $\sigma(k)$ is a smooth function of k . Fourth, alike the translog function but unlike VES production functions discussed in a wave of articles around 1970 (Lu, 1967; Sato and Hoffman, 1968; Kadiyala, 1972)² it naturally lends itself to further generalizations. For example, mirroring the extension from the Cobb–Douglas to the CES and from the CES to the IEES, the elasticity of elasticity of substitution could be made isoelastic instead of constant. One could thus eliminate one of the potential limitations of IEES functions: that $\sigma(k)$ is monotone in k .³

²See Mishra (2010) for a review of the history of production functions.

³Another issue which ought to be addressed in the future is, how to generalize IEES functions into higher dimensions. This task is, however, plagued by the fact that the elasticity of substitution is not a unique concept for functions of more than two inputs (Blackorby and Russell, 1989). Notable early

Finally, a big advantage of IEES functions is that they are readily useful for empirical applications. To justify this claim, we estimate the parameters of all four considered types of IEES production functions, with capital and labor as inputs, based on post-war US data, under three alternative estimation strategies. Upon comparison we argue that, analogously to the case of CES functions (Klump, McAdam, and Willman, 2007), most reliable estimates are obtained when using a three-equation supply-side system estimator.

In our empirical study we find that in post-war US, the elasticity of substitution σ has been systematically positively related to the capital–labor ratio in effective units, \bar{k} (i.e., after accounting for factor-augmenting technical change) as well as the capital–output ratio k/y . The null hypothesis of the CES specification is very robustly rejected. We also observe, consistently across all considered IEES functions, that the elasticity of substitution σ has been below unity on average, first fluctuating around 0.8–0.9 until the 1980s and then embarking on a secular downward trend.

From the theoretical angle, our research is also tangent to the papers which endogenize the elasticity of substitution within various general equilibrium frameworks. First, following the lead of Miyagiwa and Papageorgiou (2007), some authors have studied growth models with two-level CES production structures (Papageorgiou and Saam, 2008; Saam, 2008; Xue and Yip, 2013). This literature implies that the aggregate elasticity of substitution is a linear combination of elasticities of substitution between capital and labor in intermediate goods sectors as well as the elasticity of substitution between intermediate goods in final goods production. In equilibrium, $\sigma(k)$ can be either monotone, hump-shaped, or U-shaped in k (Xue and Yip, 2013). Second, following the lead of Jones (2005), other authors have considered frameworks with optimal technology choice at the level of firms (Growiec, 2008a,b; Matveenko and Matveenko, 2015). These are however static models where the aggregate elasticity of substitution σ , although different from the local one, does not depend on k in equilibrium. Finally, Irmen (2011) and Leon-Ledesma and Satchi (2015) have put forward dynamic models with endogenous technology choice, demonstrating how the equilibrium value of σ can evolve over time, driven by factor accumulation and technical change. In contrast to these papers, our contribution posits that the linkage between σ and k is technological, not economic.

The remainder of the article is structured as follows. Section 2 defines IEES production functions and derives their key properties. Section 3 contains a detailed elaboration of three cases of IEES functions: where the elasticity of substitution is isoelastic with respect to the relative factor share, the marginal rate of substitution, and the factor ratio k . Section 4 complements the analysis with the *capital deepening* representation of the production function (Klenow and Rodriguez-Clare, 1997; Madsen, 2010) and elabo-

contributions in this vein have been due to Gorman (1965) and Hanoch (1971).

rates on an IEES function where the elasticity of substitution is isoelastic with respect to the degree of capital deepening, k/y . Section 5 discusses the role of factor-augmenting technical change with IEES production. Section 6 illustrates the usefulness of IEES production functions in empirical applications by applying the framework to post-war US data. Section 7 concludes. The description of our dataset as well as some robustness checks of the empirical exercise are relegated to the appendix.

2 Definitions and Construction

For any constant-returns-to-scale (CRS) production function F of two inputs, K and L , one can write $Y = F(K, L)$ in its intensive form $y = f(k)$, where $y = Y/L$ and $k = K/L$. We assume that $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is three times continuously differentiable, increasing and concave in its whole domain.⁴

All the analysis will be carried out in *normalized units*. While generally redundant for Cobb–Douglas production functions due to their multiplicative character, it has been shown for the case of CES functions (de La Grandville, 1989; Klump and de La Grandville, 2000) that production function normalization is crucial for obtaining clean identification of the role of each of its parameters. As we shall see shortly, the same argument applies equally forcefully to the proposed class of IEES functions.

The natural objects of comparison in the current study are the Cobb–Douglas and the CES production function with constant returns to scale. The normalized Cobb–Douglas function is written as:

$$y = f(k) = y_0 \left(\frac{k}{k_0} \right)^{\pi_0}, \quad k_0, y_0 > 0, \pi_0 \in (0, 1). \quad (1)$$

The normalized CES production function is, in turn:

$$y = f(k) = y_0 \left(\pi_0 \left(\frac{k}{k_0} \right)^{\frac{\sigma-1}{\sigma}} + (1 - \pi_0) \right)^{\frac{\sigma}{\sigma-1}}, \quad k_0, y_0 > 0, \pi_0 \in (0, 1), \sigma > 0, \quad (2)$$

converging to the Cobb–Douglas function as the elasticity of substitution $\sigma \rightarrow 1$, to a linear function as $\sigma \rightarrow +\infty$, and to a Leontief (minimum) function as $\sigma \rightarrow 0_+$.

The following elementary concepts are central to our analysis:

- **Factor shares.** The partial elasticity of output Y with respect to K is defined as $\pi(k) = \frac{kf'(k)}{f(k)} \in [0, 1]$. If markets are perfectly competitive, this elasticity is also equal to the capital's share of output, $\frac{rk}{y}$. By constant returns to scale, implying that the labor share is $1 - \pi(k)$, it is also easily obtained that the ratio of factor

⁴Allowing K and L to be expressed in effective, technology-adjusted units is relegated to Section 5. At this point, it suffices to mention that all our results remain unchanged.

shares (and of partial elasticities), strictly increasing in $\pi(k)$, is equal to $\frac{\pi(k)}{1-\pi(k)} = \frac{kf'(k)}{f(k)-kf'(k)} \geq 0$.

The Cobb–Douglas production function is characterized by constant factor shares, with $\pi(k) \equiv \pi_0$ for all $k \geq 0$. For the CES production function, the ratio of factor shares $\frac{\pi(k)}{1-\pi(k)} = \frac{\pi_0}{1-\pi_0} \left(\frac{k}{k_0}\right)^{\frac{\sigma-1}{\sigma}}$ increases with k , from 0 when $k = 0$ to $+\infty$ as $k \rightarrow \infty$, if $\sigma > 1$. Conversely, if $\sigma < 1$ then the ratio gradually declines, from $+\infty$ towards 0.

- **Marginal rate of substitution (MRS).** For constant-returns-to-scale functions of two inputs, the MRS – capturing the slope of the isoquant – is computed as $MRS(k) \equiv \varphi(k) = -\frac{1-\pi(k)}{\pi(k)}k = -\frac{f(k)}{f'(k)} + k \leq 0$. If markets are perfectly competitive, the MRS is also equal to minus the relative price of labor as compared to capital, $\frac{w}{r} = \frac{1-\pi(k)}{\pi(k)}k = -\varphi(k)$. Monotonicity and concavity of the production function f imply that the MRS is negative and (at least weakly) declines with k .

The Cobb–Douglas function has a linearly declining MRS $\varphi(k) = \varphi_0 \left(\frac{k}{k_0}\right)$.⁵ The CES function, in turn, has an isoelastic MRS $\varphi(k) = \varphi_0 \left(\frac{k}{k_0}\right)^{1/\sigma}$. In both cases, MRS unambiguously declines from 0 when $k = 0$ to $-\infty$ when $k \rightarrow \infty$.

- **Elasticity of substitution.** The elasticity of substitution – measuring the curvature of the isoquant, i.e., the elasticity of changes in the factor ratio k in reaction to changes in the MRS – is computed as $\sigma(k) = \frac{\varphi(k)}{k\varphi'(k)} = -\frac{f'(k)(f(k)-kf'(k))}{kf(k)f''(k)} \geq 0$. Concavity of the production function f implies that the elasticity of substitution is non-negative.

The Cobb–Douglas function implies $\sigma(k) \equiv 1$ for all $k \geq 0$. For CES functions, the elasticity of substitution $\sigma > 0$ is a constant parameter.

The following definitions are central to this paper.

Definition 1 *The elasticity of elasticity of substitution with respect to x , $EES(x)$, is defined as the elasticity with which the elasticity of substitution σ reacts to changes in x :*

$$EES(x) = \frac{\partial \sigma(x)}{\partial x} \frac{x}{\sigma(x)} = \frac{\sigma'(k)}{\sigma(k)} \frac{x(k)}{x'(k)}, \quad (3)$$

where the last equality assumes that x is a differentiable function of k . We consider four argu-

⁵Using the notation $\varphi_0 = -\frac{1-\pi_0}{\pi_0}k_0 < 0$.

ments of the EES:

$$EES \left(\frac{\pi}{1-\pi} \right) = \frac{\pi(k)(1-\pi(k)) \sigma'(k)}{\pi'(k) \sigma(k)}, \quad (4)$$

$$EES(\varphi) = \frac{\varphi(k) \sigma'(k)}{\varphi'(k) \sigma(k)} = k\sigma'(k), \quad (5)$$

$$EES(k) = \frac{k\sigma'(k)}{\sigma(k)}, \quad (6)$$

$$EES \left(\frac{k}{y} \right) = \frac{k}{1-\pi(k)} \frac{\sigma'(k)}{\sigma(k)}. \quad (7)$$

Definition 2 The isoelastic elasticity of substitution production function $IEES(x)$ is a function for which $EES(x) \equiv \text{const}$.

In what follows, we shall characterize the four respective IEES functions, with $x \in \left\{ \frac{\pi}{1-\pi}; \varphi; k; \frac{k}{y} \right\}$.⁶ Please observe that for every CES or Cobb–Douglas function with a constant σ , $EES(x) = 0$ for all x , and thus they naturally belong to the wider IEES class as well. Another observation is that EES is a *third-order characteristic* of any function f : existence of $\sigma'(k)$ for all k requires that f is at least three times differentiable in its domain. Standard axioms of production functions do not place any sign restrictions on $f^{(3)}(k)$ and thus on EES , a degree of freedom that we shall exploit.

We are now in the position to spell out the main results of the current study.

Construction. The construction of a function f whose elasticity of substitution $\sigma(k)$ is of given form can be obtained in two steps: in the first step, $\sigma(k)$ is integrated up to yield the marginal rate of substitution $\varphi(k)$; in the second step $\varphi(k)$ is integrated up to yield the function $f(k)$ itself.⁷ Formally,

$$\sigma(k) = \frac{\varphi(k)}{k\varphi'(k)} \Rightarrow \varphi(k) = -\exp \left(\int \frac{dk}{k\sigma(k)} \right), \quad (8)$$

$$\varphi(k) = -\frac{f(k)}{f'(k)} + k \Rightarrow f(k) = \exp \left(\int \frac{dk}{k - \varphi(k)} \right). \quad (9)$$

Both constants of integration have to be picked specifically to maintain production function normalization. For IEES production functions, integration (8) can be executed easily, yielding closed, economically interpretable formulas for the MRS as a function of k . In contrast, integration (9) generally cannot be performed in elementary functions – but for a few notable exceptions, some of which have already been discussed in the literature.

⁶The last case requires also a more general elaboration of the *capital deepening* production function representation, i.e. rewriting $y = f(k)$ in the form of $y = h(k/y)$.

⁷Solving it in a single step is also possible but requires solving a second-order nonlinear differential equation.

3 Properties of IEES Functions

3.1 The IEES $\left(\frac{\pi}{1-\pi}\right)$ Function

The IEES $\left(\frac{\pi}{1-\pi}\right)$ production function, defined as a function for which $EES\left(\frac{\pi}{1-\pi}\right) = \psi$, where $\psi \in \mathbb{R}$ is a constant, implies (upon normalization) that the elasticity of substitution follows:

$$\frac{\sigma}{\sigma_0} = \left(\frac{\pi}{1-\pi} \frac{1-\pi_0}{\pi_0} \right)^\psi. \quad (10)$$

In this case, integration (8) yields the following formula for the MRS:

$$\varphi(k) = \varphi_0 \left(\frac{1}{\sigma_0} \left(\frac{k}{k_0} \right)^{-\psi} + \left(1 - \frac{1}{\sigma_0} \right) \right)^{-\frac{1}{\psi}}, \quad (11)$$

where $\varphi_0 = -\left(\frac{1-\pi_0}{\pi_0}\right) k_0$.

Hence, the relative factor share satisfies:

$$\frac{\pi}{1-\pi} = \frac{\pi_0}{1-\pi_0} \left(\frac{1}{\sigma_0} + \left(1 - \frac{1}{\sigma_0} \right) \left(\frac{k}{k_0} \right)^\psi \right)^{\frac{1}{\psi}}. \quad (12)$$

Both above formulas demonstrate the symmetry, owing to which the IEES $\left(\frac{\pi}{1-\pi}\right)$ function is an equally natural generalization of the CES as the CES is a generalization of the Cobb–Douglas (isoelastic) production function. For the CES function, the MRS and relative factor shares are Cobb–Douglas (isoelastic) functions of k and the elasticity of substitution is constant. For the IEES $\left(\frac{\pi}{1-\pi}\right)$ function, the MRS and relative factor share are CES functions of k and the elasticity of substitution is Cobb–Douglas (isoelastic) in the relative factor share.

Inserting (12) back into (10) implies that the elasticity of substitution is the following function of k :

$$\sigma(k) = 1 + (\sigma_0 - 1) \left(\frac{k}{k_0} \right)^\psi, \quad (13)$$

and hence $\sigma(k) > 1$ for all k if $\sigma_0 > 1$, irrespective of the value of ψ , and conversely, $\sigma(k) < 1$ for all k if $\sigma_0 < 1$. Hence, perhaps disappointingly, capital and labor are either always gross substitutes or always gross complements here. Due to the strict monotonicity of the relative factor share with respect to k (equation (10)), the elasticity of substitution $\sigma(k)$ cannot cross unity. Moreover, the case $\sigma_0 = 1$ automatically reduces the IEES $\left(\frac{\pi}{1-\pi}\right)$ function directly to the Cobb–Douglas specification.

To further illustrate the properties of the current production function specification, we shall consider four specific cases, delineated by the assumptions made with respect to ψ and σ_0 . We shall also discuss the special cases with $\psi = \pm 1$ for which integration (9) yields known closed-form formulas.⁸ The case $\psi = 1$ corresponds to the “variable

⁸Symbolic integration reveals that closed-form formulas (albeit huge and generally difficult to interpret) exist also for $\psi = \pm 2, \pm \frac{1}{2}, -3$. They are available from the author upon request.

elasticity of substitution" (VES) production function due to [Revankar \(1971\)](#) whereas the case $\psi = -1$ captures the Stone–Geary production function.

Case with $\psi > 0$ and $\sigma_0 > 1$. In this case, factors of production are always gross substitutes and hence the capital share increases with the capital–labor ratio k . Since also the elasticity of substitution increases with the capital share, it follows that the elasticity of substitution increases with k as well. The production function is well-defined, increasing and concave in its entire domain $k \in [0, +\infty)$. We obtain the following limits:

$$\lim_{k \rightarrow 0} \frac{\pi(k)}{1 - \pi(k)} = \frac{\pi_0}{1 - \pi_0} \sigma_0^{-\frac{1}{\psi}} > 0, \quad \lim_{k \rightarrow \infty} \frac{\pi(k)}{1 - \pi(k)} = +\infty, \quad (14)$$

$$\lim_{k \rightarrow 0} \varphi(k) = 0, \quad \lim_{k \rightarrow \infty} \varphi(k) = \varphi_0 \left(\frac{\sigma_0}{\sigma_0 - 1} \right)^{\frac{1}{\psi}} < 0, \quad (15)$$

$$\lim_{k \rightarrow 0} \sigma(k) = 1, \quad \lim_{k \rightarrow \infty} \sigma(k) = +\infty. \quad (16)$$

Case with $\psi < 0$ and $\sigma_0 > 1$. In this case, factors of production are always gross substitutes and hence the capital share increases with the capital–labor ratio k . Since the elasticity of substitution, on the other hand, decreases with the capital share, it follows that the elasticity of substitution decreases with k as well. The production function is well-defined, increasing and concave in its entire domain $k \in [0, +\infty)$. We obtain the following limits:

$$\lim_{k \rightarrow 0} \frac{\pi(k)}{1 - \pi(k)} = 0, \quad \lim_{k \rightarrow \infty} \frac{\pi(k)}{1 - \pi(k)} = \frac{\pi_0}{1 - \pi_0} \sigma_0^{-\frac{1}{\psi}} > 0, \quad (17)$$

$$\lim_{k \rightarrow 0} \varphi(k) = \varphi_0 \left(\frac{\sigma_0}{\sigma_0 - 1} \right)^{\frac{1}{\psi}} < 0, \quad \lim_{k \rightarrow \infty} \varphi(k) = -\infty, \quad (18)$$

$$\lim_{k \rightarrow 0} \sigma(k) = +\infty, \quad \lim_{k \rightarrow \infty} \sigma(k) = 1. \quad (19)$$

Case with $\psi > 0$ and $\sigma_0 < 1$. In this case, factors of production are always gross complements and hence the capital share is inversely related to the capital–labor ratio k . Since the elasticity of substitution, on the other hand, increases with the capital share, it follows that the elasticity of substitution falls with k . The production function is well-defined, increasing and concave only for $k \in [0, k_{max}]$, where $k_{max} = k_0(1 - \sigma_0)^{-1/\psi}$. We obtain the following limits:

$$\lim_{k \rightarrow 0} \frac{\pi(k)}{1 - \pi(k)} = \frac{\pi_0}{1 - \pi_0} \sigma_0^{-\frac{1}{\psi}} > 0, \quad \lim_{k \rightarrow k_{max}} \frac{\pi(k)}{1 - \pi(k)} = 0, \quad (20)$$

$$\lim_{k \rightarrow 0} \varphi(k) = 0, \quad \lim_{k \rightarrow k_{max}} \varphi(k) = -\infty, \quad (21)$$

$$\lim_{k \rightarrow 0} \sigma(k) = 1, \quad \lim_{k \rightarrow k_{max}} \sigma(k) = 0. \quad (22)$$

Case with $\psi < 0$ and $\sigma_0 < 1$. In this case, factors of production are always gross complements and hence the capital share is inversely related to the capital–labor ratio

k . Since also the elasticity of substitution is inversely related to the capital share, it follows that the elasticity of substitution increases with k . The production function is well-defined, increasing and concave only for $k \in [k_{min}, +\infty)$, where $k_{min} = k_0(1 - \sigma_0)^{-1/\psi}$. We obtain the following limits:

$$\lim_{k \rightarrow k_{min}} \frac{\pi(k)}{1 - \pi(k)} = +\infty, \quad \lim_{k \rightarrow \infty} \frac{\pi(k)}{1 - \pi(k)} = \frac{\pi_0}{1 - \pi_0} \sigma_0^{-\frac{1}{\psi}} > 0, \quad (23)$$

$$\lim_{k \rightarrow k_{min}} \varphi(k) = 0, \quad \lim_{k \rightarrow \infty} \varphi(k) = -\infty, \quad (24)$$

$$\lim_{k \rightarrow k_{min}} \sigma(k) = 0, \quad \lim_{k \rightarrow \infty} \sigma(k) = 1. \quad (25)$$

As shown in Section 6, our empirical analysis suggests that this case of IEES($\frac{\pi}{1-\pi}$) functions is preferred by the data on aggregate production in the post-war US economy.

Revankar's VES production function. Assuming that $\psi = 1$, following [Revankar \(1971\)](#), allows us to find the antiderivative in (9) in elementary functions. The normalized "variable elasticity of substitution" (Revankar's VES) production function with constant returns to scale reads:

$$y = f(k) = y_0 \left(\frac{k}{k_0} \right)^{\frac{\pi_0}{\pi_0 + \sigma_0(1 - \pi_0)}} \left(\pi_0 \left(\frac{\sigma_0 - 1}{\sigma_0} \right) \left(\frac{k}{k_0} \right) + \frac{\pi_0 + \sigma_0(1 - \pi_0)}{\sigma_0} \right)^{\frac{\sigma_0(1 - \pi_0)}{\pi_0 + \sigma_0(1 - \pi_0)}}, \quad (26)$$

or in non-normalized notation, $f(k) = Ak^\alpha(Bk + 1)^{1-\alpha}$, with $\alpha \in (0, 1)$, $A > 0$ and $B \in \mathbb{R}$. Please observe the domain restriction $k \leq -1/B$ which is in force if $B < 0$ (i.e., $\sigma_0 < 1$).

It is notable that while several of the production functions derived around 1970, which do not belong to the class of IEES functions, have remained something of a theoretical curiosity, the Revankar's VES function has been repeatedly used in empirical studies, even quite recently ([Karagiannis, Palivos, and Papageorgiou, 2005](#)).

Stone–Geary production function. Assuming that $\psi = -1$ also allows us to find the antiderivative in (9) in elementary functions. The normalized Stone–Geary production function (i.e., Cobb–Douglas production function of a shifted input) is:

$$y = f(k) = y_0 \left(\left(\frac{k}{k_0} \right) \left(\frac{\sigma_0 \pi_0 + (1 - \pi_0)}{\sigma_0} \right) + (1 - \pi_0) \frac{\sigma_0 - 1}{\sigma_0} \right)^{\frac{\sigma_0 \pi_0}{\sigma_0 \pi_0 + (1 - \pi_0)}}, \quad (27)$$

or in non-normalized notation, $f(k) = A(k + B)^\alpha$, with $\alpha \in (0, 1)$, $A > 0$ and $B \in \mathbb{R}$. Please observe the domain restriction $k \geq -B$ which is in force if $B < 0$ (i.e., $\sigma_0 < 1$).

3.2 The IEES(MRS) Function

The IEES(MRS) production function, defined as a function for which $EES(\varphi) = \psi$, where $\psi \in \mathbb{R}$ is a constant, implies (upon normalization) that the elasticity of substitution follows:

$$\frac{\sigma}{\sigma_0} = \left(\frac{\varphi}{\varphi_0} \right)^\psi, \quad (28)$$

where $\varphi_0 = - \left(\frac{1-\pi_0}{\pi_0} \right) k_0$.

In this case, integration (8) yields the following formula for the MRS:

$$\varphi(k) = \varphi_0 \left(1 + \frac{\psi}{\sigma_0} \ln \left(\frac{k}{k_0} \right) \right)^{\frac{1}{\psi}}. \quad (29)$$

Hence, the relative factor share satisfies:

$$\frac{\pi}{1-\pi} = \frac{\pi_0}{1-\pi_0} \frac{k}{k_0} \left(1 + \frac{\psi}{\sigma_0} \ln \left(\frac{k}{k_0} \right) \right)^{-\frac{1}{\psi}} \quad (30)$$

Inspection of the above formulas reveals that the MRS is a logarithmic function of k . The relative factor share is, on the other hand, a product of a logarithmic and a linear function of k . As opposed to the cases of the Cobb–Douglas, CES, and IEES($\frac{\pi}{1-\pi}$) functions, relative factor shares are no longer a monotonic function of k . There exists a *unique point of reversal*, coinciding with the point where the elasticity of substitution crosses unity, $\tilde{k} = k_0 e^{-\frac{\sigma_0-1}{\psi}}$ with $\sigma(\tilde{k}) = 1$.

Inserting (30) back into (28) implies that the elasticity of substitution is the following function of k :

$$\sigma(k) = \sigma_0 + \psi \ln \left(\frac{k}{k_0} \right). \quad (31)$$

To further illustrate the properties of the current production function specification, we shall consider two specific cases, delineated by the assumptions made with respect to ψ . Unfortunately, to our knowledge, IEES(MRS) functions cannot be obtained in a closed form.

Case with $\psi > 0$. In this case, the elasticity of substitution decreases with the marginal rate of substitution ($\varphi_0 < 0$) and thus increases with the factor ratio k (recall that by concavity and constant returns to scale, the MRS necessarily decreases with k). The production function is well-defined, increasing and concave only for $k \in [k_{min}, +\infty)$, where $k_{min} = k_0 e^{-\sigma_0/\psi}$. The relative factor share $\frac{\pi(k)}{1-\pi(k)}$ (and thus the capital's share $\pi(k)$ as well) follows a non-monotonic pattern with k , declining if $k \in (k_{min}, \tilde{k})$ and increasing for $k > \tilde{k}$. The minimum capital share, obtained at the point \tilde{k} , is equal to:

$$\left(\frac{\pi}{1-\pi} \right)_{min} = \frac{\pi(\tilde{k})}{1-\pi(\tilde{k})} = \frac{\pi_0}{1-\pi_0} e^{-\frac{\sigma_0-1}{\psi}} \sigma_0^{\frac{1}{\psi}}. \quad (32)$$

We also obtain the following limits:

$$\lim_{k \rightarrow k_{min}} \frac{\pi(k)}{1 - \pi(k)} = +\infty, \quad \lim_{k \rightarrow \infty} \frac{\pi(k)}{1 - \pi(k)} = +\infty, \quad (33)$$

$$\lim_{k \rightarrow k_{min}} \varphi(k) = 0, \quad \lim_{k \rightarrow \infty} \varphi(k) = -\infty, \quad (34)$$

$$\lim_{k \rightarrow k_{min}} \sigma(k) = 0, \quad \lim_{k \rightarrow \infty} \sigma(k) = +\infty. \quad (35)$$

As shown in Section 6, our empirical analysis suggests that this case of IEES(MRS) functions is preferred by the data on aggregate production in the post-war US economy. We also find $\sigma_0 < 1$.

Case with $\psi < 0$. In this case, the elasticity of substitution increases with the marginal rate of substitution and thus falls with the factor ratio k . The production function is well-defined, increasing and concave only for $k \in [0, k_{max}]$, where $k_{max} = k_0 e^{-\sigma_0/\psi}$. The relative factor share $\frac{\pi(k)}{1-\pi(k)}$ (and thus the capital's share $\pi(k)$ as well) follows a non-monotonic pattern with k , increasing when $k \in (0, \tilde{k})$ and falling for $k \in (\tilde{k}, k_{max})$. The maximum capital share, obtained at the point \tilde{k} , is equal to:

$$\left(\frac{\pi}{1 - \pi} \right)_{max} = \frac{\pi(\tilde{k})}{1 - \pi(\tilde{k})} = \frac{\pi_0}{1 - \pi_0} e^{-\frac{\sigma_0 - 1}{\psi}} \sigma_0^{\frac{1}{\psi}}. \quad (36)$$

We also obtain the following limits:

$$\lim_{k \rightarrow 0} \frac{\pi(k)}{1 - \pi(k)} = 0, \quad \lim_{k \rightarrow k_{max}} \frac{\pi(k)}{1 - \pi(k)} = 0, \quad (37)$$

$$\lim_{k \rightarrow 0} \varphi(k) = 0, \quad \lim_{k \rightarrow k_{max}} \varphi(k) = -\infty, \quad (38)$$

$$\lim_{k \rightarrow 0} \sigma(k) = +\infty, \quad \lim_{k \rightarrow k_{max}} \sigma(k) = 0. \quad (39)$$

3.3 The IEES(k) Function

The IEES(k) production function, defined as a function for which $EES(k) = \psi$, where $\psi \in \mathbb{R}$ is a constant, implies (upon normalization) that the elasticity of substitution follows:

$$\frac{\sigma}{\sigma_0} = \left(\frac{k}{k_0} \right)^\psi. \quad (40)$$

In this case, integration (8) yields the following formula for the MRS:

$$\varphi(k) = \varphi_0 e^{\frac{1}{\psi\sigma_0} \left(1 - \left(\frac{k}{k_0} \right)^{-\psi} \right)}, \quad (41)$$

where $\varphi_0 = - \left(\frac{1 - \pi_0}{\pi_0} \right) k_0$.

Hence, the relative factor share satisfies:

$$\frac{\pi}{1 - \pi} = \frac{\pi_0}{1 - \pi_0} \frac{k}{k_0} e^{-\frac{1}{\psi\sigma_0} \left(1 - \left(\frac{k}{k_0} \right)^{-\psi} \right)} \quad (42)$$

Inspection of the above formulas reveals that the MRS is an exponential function of k . Relative factor shares are, on the other hand, a product of an exponential and a linear function of k . As opposed to the cases of the Cobb–Douglas, CES, and IEES($\frac{\pi}{1-\pi}$) functions, and alike the IEES(MRS) function, the relative factor share is a non-monotonic function of k . There is a *unique point of reversal*, coinciding with the point where the elasticity of substitution crosses unity, $\tilde{k} = k_0\sigma_0^{-1/\psi}$ with $\sigma(\tilde{k}) = 1$.

To further illustrate the properties of the current production function specification, we shall consider two specific cases, delineated by the assumptions made with respect to ψ . Unfortunately, to our knowledge, IEES(k) functions cannot be obtained in a closed form.

Case with $\psi > 0$. In this case, we assume that the elasticity of substitution increases with the factor ratio k . The production function is well-defined, increasing and concave in its domain $k \in [0, +\infty)$. The relative factor share $\frac{\pi(k)}{1-\pi(k)}$ (and thus the capital's share $\pi(k)$ as well) follows a non-monotonic pattern with k , declining if $k \in (0, \tilde{k})$ and increasing for $k > \tilde{k}$. The minimum capital share, obtained at the point \tilde{k} , is equal to:

$$\left(\frac{\pi}{1-\pi}\right)_{min} = \frac{\pi(\tilde{k})}{1-\pi(\tilde{k})} = \frac{\pi_0}{1-\pi_0} e^{-\frac{\sigma_0-1}{\psi\sigma_0}} \sigma_0^{-\frac{1}{\psi}}. \quad (43)$$

We also obtain the following limits:

$$\lim_{k \rightarrow 0} \frac{\pi(k)}{1-\pi(k)} = +\infty, \quad \lim_{k \rightarrow \infty} \frac{\pi(k)}{1-\pi(k)} = +\infty, \quad (44)$$

$$\lim_{k \rightarrow 0} \varphi(k) = 0, \quad \lim_{k \rightarrow \infty} \varphi(k) = \varphi_0 e^{\frac{1}{\psi\sigma_0}} < 0, \quad (45)$$

$$\lim_{k \rightarrow 0} \sigma(k) = 0, \quad \lim_{k \rightarrow \infty} \sigma(k) = +\infty. \quad (46)$$

As shown in Section 6, our empirical analysis suggests that this case of IEES(k) functions is preferred by the data on aggregate production in the post-war US economy. We also find $\sigma_0 < 1$.

Case with $\psi < 0$. In this case, we assume that the elasticity of substitution decreases with the factor ratio k . The production function is well-defined, increasing and concave in its domain $k \in [0, +\infty)$. The relative factor share $\frac{\pi(k)}{1-\pi(k)}$ (and thus the capital's share $\pi(k)$ as well) follows a non-monotonic pattern with k , increasing when $k \in (0, \tilde{k})$ and falling for $k > \tilde{k}$. The maximum capital share, obtained at the point \tilde{k} , is equal to:

$$\left(\frac{\pi}{1-\pi}\right)_{max} = \frac{\pi(\tilde{k})}{1-\pi(\tilde{k})} = \frac{\pi_0}{1-\pi_0} e^{-\frac{\sigma_0-1}{\psi\sigma_0}} \sigma_0^{-\frac{1}{\psi}}. \quad (47)$$

We also obtain the following limits:

$$\lim_{k \rightarrow 0} \frac{\pi(k)}{1 - \pi(k)} = 0, \quad \lim_{k \rightarrow \infty} \frac{\pi(k)}{1 - \pi(k)} = 0, \quad (48)$$

$$\lim_{k \rightarrow 0} \varphi(k) = \varphi_0 e^{\frac{1}{\psi \sigma_0}} < 0, \quad \lim_{k \rightarrow \infty} \varphi(k) = -\infty, \quad (49)$$

$$\lim_{k \rightarrow 0} \sigma(k) = +\infty, \quad \lim_{k \rightarrow \infty} \sigma(k) = 0. \quad (50)$$

4 The Capital Deepening Production Function Representation and the IEES(k/y) Function

It is popular, especially in the growth and development accounting literature (see e.g., [Klenow and Rodriguez-Clare, 1997](#); [Madsen, 2010](#)), to rewrite the aggregate production function so that it takes the capital–output ratio $\kappa \equiv K/Y = k/y$ instead of k as its input. Increases in κ are then identified with *capital deepening*. The key reason for making such a transformation is that, unlike k , the capital deepening term κ should not exhibit a strong upward trend, and dealing with variables without discernible trends has its documented statistical advantages. And indeed, relative stability of the capital–output ratio (one of the “great ratios” in macroeconomics) has been long taken as a stylized fact, together with relative stability of factor shares. Only relatively recently have both postulates been questioned; still, if y and k exhibit upward trends, by definition k/y *must* be at least growing much slower than k , underscoring the empirical value of the current representation.

As a preliminary remark, observe how easy it is to rewrite the normalized Cobb–Douglas and CES functions with constant returns to scale in the capital deepening form:

$$y = y_0 \left(\frac{\kappa}{\kappa_0} \right)^{\frac{\pi_0}{1-\pi_0}}, \quad \kappa_0, y_0 > 0, \pi_0 \in (0, 1), \quad (51)$$

$$y = y_0 \left(\frac{1}{1-\pi_0} - \frac{\pi_0}{1-\pi_0} \left(\frac{\kappa}{\kappa_0} \right)^{\frac{\sigma-1}{\sigma}} \right)^{-\frac{\sigma}{\sigma-1}}, \quad \kappa_0, y_0 > 0, \pi_0 \in (0, 1), \sigma > 0. \quad (52)$$

The implied relative factor shares $\Pi(\kappa) \equiv \frac{\pi(\kappa)}{1-\pi(\kappa)}$ are, respectively, equal to $\Pi(\kappa) = \frac{\pi_0}{1-\pi_0}$ (a constant) in the Cobb–Douglas case, and

$$\Pi(\kappa) = \frac{\frac{\pi_0}{1-\pi_0} \left(\frac{\kappa}{\kappa_0} \right)^{\frac{\sigma-1}{\sigma}}}{\frac{1}{1-\pi_0} - \frac{\pi_0}{1-\pi_0} \left(\frac{\kappa}{\kappa_0} \right)^{\frac{\sigma-1}{\sigma}}}, \quad \pi(\kappa) = \pi_0 \left(\frac{\kappa}{\kappa_0} \right)^{\frac{\sigma-1}{\sigma}}, \quad (53)$$

in the CES case. Hence, the capital share is isoelastic in the degree of capital deepening κ and increases with κ if and only if $\sigma > 1$, i.e., if capital and labor are gross substitutes.

Finally, observe that the functional form of equation (53) does not by itself preclude cases with $\pi(\kappa) > 1$. These cases are made impossible only by the range of the CES function which restricts the support of $\kappa = k/y$ appropriately.

Although for arbitrary (increasing and concave) production functions, rewriting them (and their implied elasticities) in terms of κ is not so easy anymore, it can always be done. Let us now recall some known relevant results.

Existence. Any increasing, concave, and constant-returns-to-scale (CRS) production function of two inputs, $Y = F(K, L)$, can be rewritten as $F\left(\frac{K}{Y}, \frac{L}{Y}\right) = 1$. Then, by the implicit function theorem,⁹ there exists a function $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\frac{L}{Y} = \frac{1}{h(K/Y)}$ and thus $y = h(\kappa)$. Note that due to concavity of F , the capital deepening term κ is always increasing in k . We also observe that the relative factor share can be computed directly as the elasticity of $h(\kappa)$ with respect to κ :

$$\Pi(\kappa) = \frac{\pi(\kappa)}{1 - \pi(\kappa)} = \frac{h'(\kappa)\kappa}{h(\kappa)}. \quad (54)$$

The existence of an *explicit* form of the function $h(\kappa)$, however, hinges on the requirement that $F(\kappa, 1/y) = 1$ can be solved for y explicitly, which need not be the case even if the functional form of F is given. Notably, it *cannot* be done for IEES functions whose explicit form is not known.¹⁰

Construction. Using this notation, the proposed two-step method for finding functions whose elasticity of substitution is given as a predefined function of the degree of capital deepening κ is as follows:

$$\sigma(\kappa) = \frac{1}{1 - \frac{\Pi'(\kappa)\kappa}{\Pi(\kappa)(1+\Pi(\kappa))}} \Rightarrow \Pi(\kappa) = \frac{1}{\exp\left(-\int \frac{\sigma(\kappa)-1}{\kappa\sigma(\kappa)} d\kappa\right) - 1}, \quad (55)$$

$$\Pi(\kappa) = \frac{h'(\kappa)\kappa}{h(\kappa)} \Rightarrow h(\kappa) = \exp\left(\int \frac{\Pi(\kappa)}{\kappa} d\kappa\right). \quad (56)$$

Unfortunately, the integrals (55)–(56) can be computed in elementary functions only for a very narrow set of functional specifications of $\sigma(\kappa)$.

Still, this apparatus enables us to define and characterize the IEES(κ) production function whose elasticity of substitution is isoelastic in the degree of capital deepening.

The IEES(κ) production function, defined as a function for which $EES(\kappa) = \psi$, where $\psi \in \mathbb{R}$ is a constant, implies (upon normalization) that the elasticity of substitution follows:

$$\frac{\sigma}{\sigma_0} = \left(\frac{\kappa}{\kappa_0}\right)^\psi = \left(\frac{k y_0}{y k_0}\right)^\psi. \quad (57)$$

⁹Which can be used because F is increasing and concave in its entire domain.

¹⁰It can be done for the special cases of Revankar's VES and Stone-Geary production function, though. Details are available upon request.

In this case, integration (55) yields the following formula for the relative factor share:

$$\Pi(\kappa) = \frac{\pi_0 \left(\frac{\kappa}{\kappa_0} \right)}{e^{\frac{1}{\psi\sigma_0} \left(1 - \left(\frac{\kappa}{\kappa_0} \right)^{-\psi} \right)} - \pi_0 \left(\frac{\kappa}{\kappa_0} \right)}. \quad (58)$$

Slight rearrangement of the above formula reveals that the capital share $\pi(\kappa)$ is a product of an exponential and a linear function of κ :

$$\pi(\kappa) = \pi_0 \left(\frac{\kappa}{\kappa_0} \right) e^{-\frac{1}{\psi\sigma_0} \left(1 - \left(\frac{\kappa}{\kappa_0} \right)^{-\psi} \right)}. \quad (59)$$

As opposed to the cases of the Cobb–Douglas, CES, and IEES($\frac{\pi}{1-\pi}$) functions, and alike the IEES(MRS) and IEES(k) functions, relative factor shares are a non-monotonic function of κ here (and thus, owing to the concavity of $F(K, L)$, of the capital–labor ratio k as well). There exists a *unique point of reversal, coinciding with the point where the elasticity of substitution crosses unity*, $\tilde{\kappa} = \kappa_0 \sigma_0^{-1/\psi}$ with $\sigma(\tilde{\kappa}) = 1$.

To further illustrate the properties of the current production function specification, we shall consider two specific cases, delineated by the assumptions made with respect to ψ . Unfortunately, to our knowledge, IEES(κ) functions cannot be obtained in a closed form.

Case with $\psi > 0$. In this case, we assume that the elasticity of substitution increases with the degree of capital deepening κ . Due to restrictions in the range of $F(K, L)$, the support of κ is restricted to $\kappa \in [\kappa_{min}, \kappa_{max}]$ where κ_{min} and κ_{max} are the two solutions to the equation $\pi(\kappa) = 1$. The capital's share $\pi(\kappa)$ follows a non-monotonic pattern with κ , declining if $\kappa \in (\kappa_{min}, \tilde{\kappa})$ and increasing for $\kappa \in (\tilde{\kappa}, \kappa_{max})$. The minimum capital share, obtained at the point $\tilde{\kappa}$, is equal to:

$$\pi_{min} = \pi(\tilde{\kappa}) = \pi_0 e^{-\frac{\sigma_0-1}{\psi\sigma_0} \sigma_0^{-\frac{1}{\psi}}}. \quad (60)$$

As shown in Section 6, our empirical analysis suggests that this case of IEES(κ) functions is preferred by the data on aggregate production in the post-war US economy. We also find $\sigma_0 < 1$.

Case with $\psi < 0$. In this case, we assume that the elasticity of substitution decreases with the degree of capital deepening κ . The capital's share $\pi(k)$ follows a non-monotonic pattern with κ , increasing when $\kappa \in (0, \tilde{\kappa})$ and falling for $\kappa > \tilde{\kappa}$. The maximum capital share, obtained at the point $\tilde{\kappa}$, is equal to:

$$\pi_{max} = \pi(\tilde{\kappa}) = \pi_0 e^{-\frac{\sigma_0-1}{\psi\sigma_0} \sigma_0^{-\frac{1}{\psi}}}, \quad (61)$$

with the following limits:

$$\lim_{\kappa \rightarrow 0} \pi(\kappa) = 0, \quad \lim_{\kappa \rightarrow \infty} \pi(\kappa) = 0. \quad (62)$$

The production function is well-defined, increasing and concave in its domain $\kappa \in [0, +\infty)$ as long as $\pi_{max} \leq 1$.

5 Factor-Augmenting Technical Change

One of the many advantages of assuming constant returns to scale lies with a clean treatment of factor-augmenting technical change. Indeed, with just a slight modification of notation, technical change can be incorporated in any CRS production function by replacing $Y = F(K, L)$ with $Y = F(\Gamma^K K, \Gamma^L L)$, or – in the intensive form – by replacing $y = f(k)$ with $\bar{y} = f(\bar{k})$, where $\bar{y} = \frac{Y}{\Gamma^L L}$ and $\bar{k} = \frac{\Gamma^K K}{\Gamma^L L}$. Crucially, owing to constant returns to scale, the functional form of f remains unchanged. And if one is ultimately interested in y instead of \bar{y} , then one may simply compute $y = \Gamma^L \bar{y} = \Gamma^L f(\bar{k}) = F(\Gamma^K k, \Gamma^L)$ after all the necessary derivations.

This last step implicitly separates the Hicks-neutral component of technical change from the *capital bias* in technical change (cf., e.g., [León-Ledesma, McAdam, and Willman, 2010](#)). This is the key insight for the current study because it allows us to define the capital share $\pi(\bar{k})$, the marginal rate of substitution $\varphi(\bar{k})$ and, crucially, the elasticity of substitution $\sigma(\bar{k})$, as a function of the capital–labor ratio *in effective units*. Hence, any capital-biased technical change (i.e., increase in Γ^K / Γ^L) acts just like physical capital accumulation, whereas labor-biased technical change (decline in Γ^K / Γ^L) affects factor shares, MRS and σ alike a decline in the capital–labor ratio k .¹¹ All functional forms remain unchanged.

Factor-augmenting technical change can be studied in the capital deepening production function representation as well. With the notation $\bar{\kappa} = \frac{\bar{k}}{\bar{y}} = \frac{\Gamma^K k}{y}$, one can easily replace $y = h(\kappa)$ with $\bar{y} = h(\bar{\kappa})$ and all the above results still go through. At the same time, this specification emphasizes that capital-augmenting technical change adds to capital deepening just like capital accumulation, while labor-augmenting technical change is neutral for capital deepening.

Clearly, both theory and data suggest that labor-augmenting technical change are likely to be dominant over the long run ([Acemoglu, 2003](#); [Klump, McAdam, and Willman, 2012](#)), and therefore the capital–labor ratio in effective units \bar{k} will likely grow slower (if at all) than the raw capital–labor ratio k . Indeed, US data include, apart from periods of growth, also prolonged periods of decline in \bar{k} . Hence, for empirical applications of IEES functions (and CES ones as well), it is important whether one considers the capital–labor ratio in effective units (\bar{k}) or just as a raw variable, measured in dollars per worker (k).

¹¹See [Growiec \(2013\)](#) for a discussion of the micro-level forces behind the direction of factor-augmenting technical change.

6 Application: The US Aggregate Production Function

Usefulness of the proposed class of IEES production functions in empirical applications follows from the fact that they provide testable predictions for the functional relationships between two observables: factor shares and the capital-labor ratio k (or the degree of capital deepening, κ). Each of the nonlinear equations (12), (30), (42) and (59) can be estimated, either separately or in a larger system, based on country-level, sectoral-level, or even firm-level data.

An important advantage of production function normalization which we use here is that each of the four IEES specifications can be used to determine simultaneously the magnitude of elasticity of substitution ψ and the *average* elasticity of substitution in the sample, σ_0 . As for the latter parameter, elasticity of substitution σ estimated under the CES specification works as a natural benchmark for comparisons.

In this section we consider an empirical application of the proposed class of IEES production functions to post-war US data. Our dataset covers the non-residential business sector of the US economy (see [Rognlie, 2015](#), for discussion) and consists of quarterly time series spanning the period 1948Q1–2013Q4. All variables are normalized around their respective geometric sample means. Detailed description of the dataset and construction of variables is included in [Appendix A.1](#).

Having sorted out the issue of factor-augmenting technical change and estimated the parameters of IEES functions, we shall study the trajectory of the elasticity of substitution between capital and labor in the US, σ_t . Thus far, the magnitude of this deep technological parameter has been consistently estimated for the US only when assuming its constancy over time (see [Klump, McAdam, and Willman, 2012](#), for a review). In contrast, our empirical investigation reveals that σ_t has exhibited substantial variation over time, and since the 1980s it has also systematically followed a downward trend.

6.1 Estimation Strategies

The parameters of IEES functions will be estimated under the baseline assumption that technological progress is exponential and purely labor-augmenting.¹² Hence, we shall assume that $\Gamma_t^K \equiv 1$ and $\Gamma_t^L = e^{\gamma_l(t-\bar{t})}$ where γ_l is the constant and exogenous rate of labor-augmenting technical change. The capital–labor ratio in effective units is then equal to:

$$\bar{k}_t = \frac{\Gamma_t^K K_t}{\Gamma_t^L L_t} = \frac{K_t}{L_t} e^{-\gamma_l(t-\bar{t})}. \quad (63)$$

¹²This assumption is consistent with a bulk of empirical literature (see the review by [Klump, McAdam, and Willman, 2012](#)). Moreover, extensive robustness analysis confirms that all our key results remain in force also when this assumption is relaxed.

The parameter \bar{t} is set such that the sample average of $\ln \bar{k}_t$ is zero. It is also observed that under purely labor-augmenting technical change, $\bar{k}_t = \kappa_t$.

We consider three alternative estimation strategies.

Single-equation NLS estimation. This strategy consists in estimating the parameters of each of the nonlinear equations (12), (30), (42) and (59) with nonlinear least squares (NLS), after taking logs. The pace of labor-augmenting technical change, γ_l , is set at 0.0045 per quarter (about 0.018 per annum)¹³ and not estimated.

The advantage of single-equation NLS estimation is that it is simple to execute and requires data solely on the relative factor share, $\pi_t/(1 - \pi_t)$, and the capital-labor ratio expressed in effective units, \bar{k}_t , or alternatively the degree of capital deepening, κ_t . The problem with this estimation method is, however, that the identification of the estimated parameters – the average elasticity of substitution σ_0 and the elasticity of elasticity of substitution ψ – based on such a scarce dataset is hard because they are deep technological constants. This argumentation is analogous to the one put forward in the CES literature which has identified the advantages of using normalized supply-side system estimation over the single-equation approach in estimating the (constant) elasticity of substitution based on time-series data (Klump, McAdam, and Willman, 2007; León-Ledesma, McAdam, and Willman, 2010). Hence, we shall also seek to estimate σ_0 and ψ jointly with π_0 and γ_l in a three-equation system, using additional data on output and relative prices.

Two-step estimation. This strategy consists in, first, estimating the parameters of a CES production function with factor-augmenting technical change ($\pi_0, \sigma_0, \gamma_l$) following the three-equation system strategy due to Klump, McAdam, and Willman (2007) and, next, assuming that the elasticity of substitution $\sigma(k)$ is given by an IEES specification and thus estimating ψ .

In the first step, the normalized supply-side system with CES production is jointly estimated:

$$\ln \left(\frac{r_t K_t}{P_t Y_t} \right) = \ln(\pi_0) + \frac{1 - \sigma_0}{\sigma_0} \left(\ln \left(\frac{Y_t K_0}{\bar{K}_t Y_0} \right) - \ln(\xi) - \ln(\Gamma_t^K) \right), \quad (64)$$

$$\ln \left(\frac{w_t L_t}{P_t Y_t} \right) = \ln(1 - \pi_0) + \frac{1 - \sigma_0}{\sigma_0} \left(\ln \left(\frac{Y_t L_0}{L_t Y_0} \right) - \ln(\xi) - \ln(\Gamma_t^L) \right), \quad (65)$$

$$\ln \left(\frac{Y_t}{Y_0} \right) = \ln(\xi) + \frac{\sigma_0}{\sigma_0 - 1} \ln \left(\pi_0 \left(\frac{K_t}{K_0} \Gamma_t^K \right)^{\frac{\sigma_0 - 1}{\sigma_0}} + (1 - \pi_0) \left(\frac{L_t}{L_0} \Gamma_t^L \right)^{\frac{\sigma_0 - 1}{\sigma_0}} \right) \quad (66)$$

where $(r_t K_t)/(P_t Y_t) = \pi_t$ and $(w_t L_t)/(P_t Y_t) = 1 - \pi_t$ stand for the capital and la-

¹³This number corresponds to our estimates of the US supply-side system which will be discussed shortly.

bor share, respectively.¹⁴ The first two equations are first-order conditions of profit maximization under perfect competition, for capital (64) and labor (65). The third equation captures the log of a CES production function (66). Residuals are allowed to be correlated across equations and therefore we use a Generalized Nonlinear Least Squares (GNLS) estimator. However, our findings are robust to alternative choices of the estimator (e.g., multivariate NLS), as well as initial values used in the estimation procedure. System estimation is expected to yield superior estimates of σ_0 (and π_0) compared to single-equation CES estimates, based uniquely on the difference between equations (64) and (65), cf. Klump, McAdam, and Willman (2007); León-Ledesma, McAdam, and Willman (2010).

In the second step, the estimate of σ_0 as well as the parameters describing deterministic factor-augmenting technical progress, obtained in the previous stage, are taken as given. It allows us to estimate the second deep parameter ψ , based on equations (12), (30), (42) and (59), with much more precision.

The advantages of two-step estimation are that (i) the properties of the first step have been thoroughly characterized in the CES literature, and that (ii) the second step is less demanding of data than in the single-equation approach. On the other hand, the disadvantage of this approach is that it inconsistently assumes the production function to be CES in the first step and IEES in the second step. This may lead to systematic errors in the case of a substantial discrepancy between both specifications (i.e., if ψ is far away from zero). Therefore we also seek to estimate ψ jointly with π_0, σ_0 and γ_l in a single system.

Joint estimation of the supply-side system with IEES production. This strategy consists in estimating all parameters of the supply-side system (64)–(66) jointly with ψ , while allowing the elasticity of substitution to be time-varying. Thus σ_0 is systematically replaced with σ_t , which itself follows the considered IEES production function:

$$\text{IEES} \left(\frac{\pi}{1-\pi} \right) : \quad \sigma_t = 1 + (\sigma_0 - 1) \left(\frac{\bar{k}_t}{\bar{k}_0} \right)^\psi, \quad (67)$$

$$\text{IEES(MRS)} : \quad \sigma_t = \sigma_0 + \psi \ln \left(\frac{\bar{k}_t}{\bar{k}_0} \right), \quad (68)$$

$$\text{IEES}(k) : \quad \sigma_t = \sigma_0 \left(\frac{\bar{k}_t}{\bar{k}_0} \right)^\psi, \quad (69)$$

$$\text{IEES}(\kappa) : \quad \sigma_t = \sigma_0 \left(\frac{\kappa_t}{\kappa_0} \right)^\psi. \quad (70)$$

There are two key advantages of joint estimation of all parameters of this system: it is consistent with the theoretical formulation of IEES production functions, and it

¹⁴The additional scaling parameter ξ is expected to be around unity, cf. Klump, McAdam, and Willman (2007), and will not play any role in our analysis.

uses a richer dataset than the single-equation NLS approach. On the other hand, it must be emphasized that after substituting any of the equations (67)–(70) into the system (64)–(66), a highly nonlinear system of equations is obtained. In particular, note, factor-augmenting technical change is embodied in the formula for the elasticity of substitution (apart from the case of the IEES(κ) production function). Identification of parameters of such a complex system can, in principle, be difficult. Fortunately, in our analysis we have not encountered any problems in this regard.¹⁵

6.2 Estimation Results

Estimation results are presented in Table 2. The top panel summarizes single-equation NLS estimates, the middle one – two-step estimates, and the bottom one – results of joint estimation of the supply-side system with IEES production. Consecutive columns pertain to the respective types of IEES functions. CES estimates are included in the first column for comparison. The results for two-step estimation of the CES function are omitted because in such a case, the first step corresponds exactly to the system approach (presented in the bottom panel) whereas the second step is void.

These results deliver three key messages. First, capital and labor are, on average, gross complements. Depending on the production function specification and estimation method, the estimated value of σ_0 ranges from 0.590 to 0.866, fully corroborating the CES results summarized by Klump, McAdam, and Willman (2012). Second, the elasticity of substitution is *not* constant over time. Across all (but one) considered specifications of IEES functions, the estimates of ψ are statistically significantly different from zero at the 1% significance level. The null of a CES production function specification is thus robustly rejected. Third, apart from single-equation NLS estimates of ψ which are likely biased (see below), we find that across all IEES specifications, the relationship between the elasticity of substitution $\sigma(\bar{k})$ and the effective capital–labor ratio \bar{k} is consistently positive, i.e., $\psi > 0$ in the cases of IEES(*MRS*), IEES(*k*) and IEES(κ), and $(\sigma_0 - 1)\psi > 0$ with $\sigma_0 < 1$ and $\psi < 0$ in the case of IEES($\frac{\pi}{1-\pi}$). In other words, we find that the more capital is accumulated per worker (in effective units), the higher is the elasticity of substitution σ .

More specifically, the elasticity of substitution estimated within the normalized supply-side system with CES production equals 0.757 (bottom panel, first column) and thus is close to the literature consensus of $\sigma_0 \approx 0.6 - 0.7$. Moreover, the result that the pace of labor-augmenting technical change equals $\gamma_l = 0.0045$ per quarter (0.018 per annum) is equally well aligned with the literature. In the second step of the two-step estimation procedure, both these numbers are taken as given; in the single-equation

¹⁵Such problems appear, however, in the robustness checks where we also allow for capital-augmenting technical change (see below).

Table 2: Summary of Baseline Estimates of IEES Production Functions

	CES	IEES($\frac{\pi}{1-\pi}$)	IEES(MRS)	IEES(\bar{k})	IEES(κ)
Single-Equation NLS ($\gamma_l = 0.0045$)					
π_0	0.327*** (0.001)	0.331*** (0.001)	0.331*** (0.001)	0.331*** (0.002)	0.324*** (0.001)
σ_0	0.613*** (0.019)	0.623*** (0.027)	0.605*** (0.021)	0.590*** (0.019)	0.745*** (0.029)
ψ		6.064*** (1.586)	-2.535*** (0.671)	-3.936*** (1.201)	2.676*** (0.726)
ADF (12), (30), (42), (59)	-2.995***	-4.156***	-4.107***	-4.030***	-2.998***
Two-Step ($\sigma_0 = 0.757$ and $\gamma_l = 0.0045$)					
π_0		0.324*** (0.001)	0.324*** (0.001)	0.325*** (0.001)	0.324*** (0.001)
ψ		-7.687*** (0.688)	2.381*** (0.540)	1.985 (1.212)	2.867*** (0.887)
ADF (67)–(70)		-3.562***	-3.256***	-3.108***	-2.979***
System Approach					
π_0	0.331*** (0.001)	0.324*** (0.001)	0.326*** (0.001)	0.326*** (0.001)	0.326*** (0.001)
σ_0	0.757*** (0.001)	0.866*** (0.018)	0.796*** (0.014)	0.793*** (0.015)	0.795*** (0.014)
ψ		-7.956*** (1.048)	0.608*** (0.228)	0.644*** (0.301)	1.032*** (0.243)
ξ	0.999*** (0.002)	1.002*** (0.002)	1.001*** (0.002)	1.001*** (0.002)	1.003*** (0.002)
γ_l	0.004*** (0.000)	0.005*** (0.000)	0.004*** (0.000)	0.004*** (0.000)	0.004*** (0.000)
ADF (64)	-2.715***	-2.805***	-2.856***	-2.830***	-2.903***
ADF (65)	-3.646***	-3.578***	-3.519***	-3.520***	-3.601***
ADF (66)	-2.494***	-2.946***	-2.971***	-2.973***	-2.977***

Notes: the superscripts ***, ** and * denote rejection of the null about parameters' insignificance at the 1%, 5% and 10% significance level, respectively. In the case of σ_0 , the null hypothesis is that $\sigma_0 = 1$ (Cobb–Douglas production). ADF stands for the Augmented Dickey-Fuller test without a constant term. The superscripts ***, ** and * in the ADF test denote rejection of the null about a unit root of the respective residuals at the 1%, 5% and 10% significance level. The number of lags for the ADF test has been determined by the BIC criterion. The numbers in parentheses are robust standard errors.

NLS procedure, only the latter one. Crucially, however, when both parameters are estimated jointly with ψ in the system approach (bottom panel, columns 2–5), they still remain in the same ballpark. Even if the estimated average elasticity of substitution may be somewhat higher than 0.7, the null of the Cobb-Douglas form is always rejected by the data. Finally, we also observe that the only specification for which we are not able to reject the null of $\psi = 0$ (CES production) is the case of the IEES(k) function, estimated in a two-step procedure. Although the standard error of estimation is quite substantial in this case, the sign of the point estimate of ψ remains in line with the results for other specifications.

Let us now clarify why we believe our single-equation NLS estimates are likely biased whereas the other ones are reliable. The reason is that simultaneous identification of two deep parameters of the production function, σ_0 and ψ , based on a single, highly nonlinear equation and two time series only (the capital–labor ratio in effective units and the relative factor share), is very demanding of the data. It is likely that the puzzling result of an opposite sign of ψ estimates is driven by short-run correlation between both variables, and thus captures cyclical co-movement rather than the underlying production technology. This conclusion is further strengthened by our robustness checks which tend to agree with our alternative, two-step and system estimates. In particular, the signs of ψ estimates under the single-equation strategy are reversed once we use quality-adjusted labor input.

The ADF statistics indicate that residuals from all estimated equations are stationary at all conventional significance levels. It is particularly important because, on the other hand, we do not find evidence for stationarity of relative factor shares.¹⁶ Viewed from the cointegration perspective, this ensures that there is no problem of spurious regression.

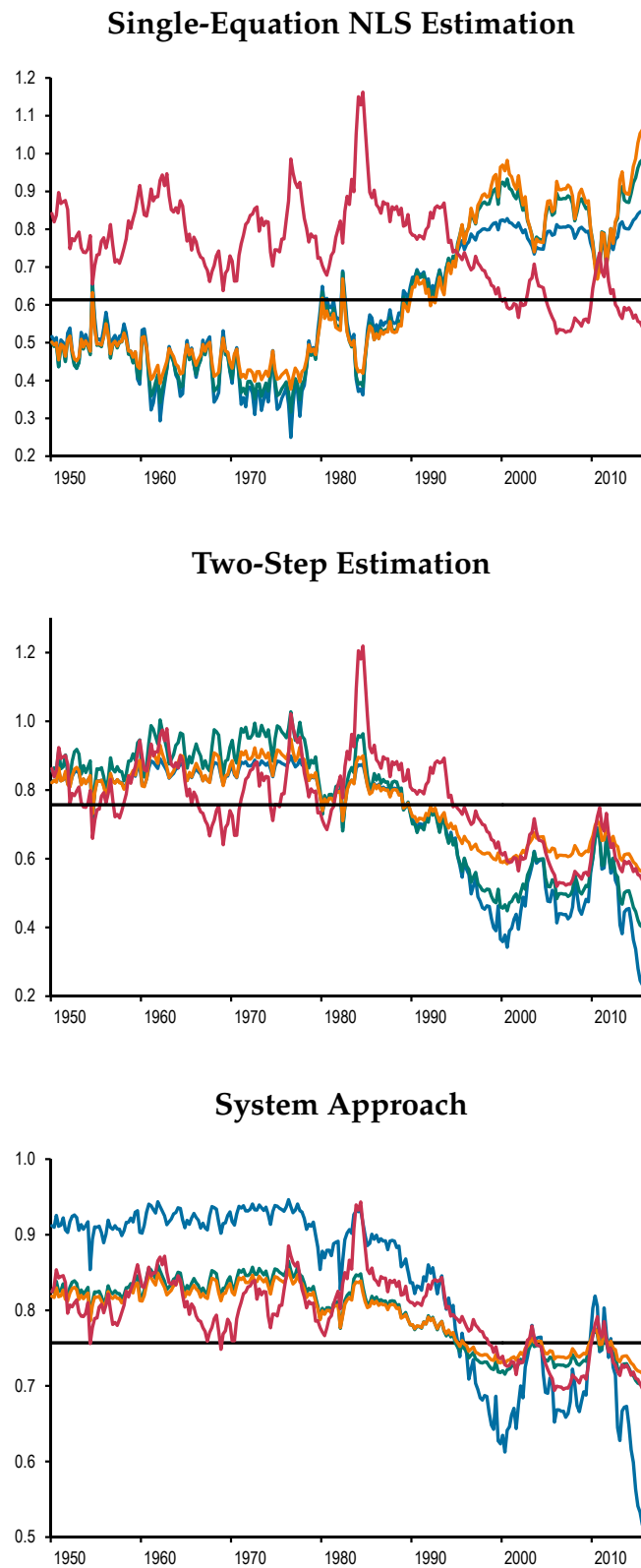
6.3 Inferring the Path of Time-Varying Elasticity of Substitution

Plugging the actual time series of the effective capital–labor ratio \bar{k}_t and the degree of capital deepening κ_t (see Figure A.1 in the Appendix) into equations (67)–(70) allows us to infer the exact time path of σ_t under each IEES specification. These paths are presented in Figure 1. All considered IEES functions are marked by different colors, whereas the horizontal black lines represent the CES benchmark.

We observe that the elasticity of substitution has exhibited substantial variability over time. The overall time pattern of σ_t , except under single-equation estimation which is likely biased, is familiar. Broadly speaking, it remained relatively stable, at

¹⁶The ADF statistic for the log relative factor share equals -1.967 and the null about a unit root cannot be rejected at the 10% significance level. This result aligns with the fact that evidence for covariance stationarity of the US labor share is weak (see Mućk, McAdam, and Growiec, 2015, for a general discussion).

Figure 1: Implied Time Paths of the Elasticity of Substitution σ_t



Notes: CES estimates, $\text{IEES}\left(\frac{\pi}{1-\pi}\right)$, $\text{IEES}(MRS)$, $\text{IEES}(k)$ and $\text{IEES}(\kappa)$.

about 0.8–0.9, from 1948 to the 1980s, after which it entered a period of secular decline. While the magnitude of the cumulative decline in σ_t since the 1980s is uncertain and depends on the assumed production function specification, the sheer existence of a downward trend in the elasticity of substitution appears to be a very robust finding. In its core, it mirrors the decline in the effective capital–labor ratio and the degree of capital deepening in the US economy (Figure A.1).

More specifically, in the case of two-step estimation (middle panel in Figure 1) the implied total decline in σ_t has been largest for the case of the IEES($\frac{\pi}{1-\pi}$) function: the post-2000 average value of σ_t equals just 0.48. The IEES(k) function is located at the other end of the spectrum: in its case, the post-2000 average of σ_t is 0.63. System estimates (bottom panel) imply somewhat smaller variability of σ_t because the absolute value of estimated ψ is lower. Except for the case of the IEES($\frac{\pi}{1-\pi}$) function, the implied elasticity of substitution has been stable around 0.8–0.85 until mid-1980s and afterwards it decreased to about 0.7. The outstanding IEES($\frac{\pi}{1-\pi}$) function implies a higher initial value of σ , about 0.9 until late 1970s, and a more pronounced decline afterwards.

6.4 Robustness Checks

As a robustness check of the previous results, we have modified our assumptions on technical progress as well as used additional data.

Firstly, we have relaxed the restriction of exponential labor-augmenting technical change by introducing a more general and flexible Box-Cox technology term (following Klump, McAdam, and Willman, 2007). Table A.1 confirms the robustness of our baseline estimates to this change. Puzzlingly, the estimated curvature parameter λ_l is slightly (but statically significantly) above unity, suggesting that labor-augmenting technical change has been *accelerating* throughout the sample. Nevertheless, the estimated parameter ψ has the same sign as in the baseline setting and is statistically significant in all specifications. Apart from single-equation NLS estimation strategy, this exercise also replicates the secular decline in the elasticity of substitution since the 1980s (see the top panel of Figure A.2).

Secondly, we have considered an alternative measure of the labor input. Indeed, total employment (employees plus the self-employed) or aggregate hours might in fact be a poor proxy of the actual flow of labor services in the economy because they ignore the ongoing changes in labor composition, or “quality”. Therefore, quality-adjusted aggregate hours due to Fernald (2012) have been used as our measure of the labor input in this robustness check (see Appendix A.1 for a detailed description). Table A.2 confirms that all our previous empirical findings are robust to this change as well. Interestingly, signs of all estimated parameters are now consistent across all estimation strategies (the inconsistency in the single-equation NLS case disappears), whereas the

implied trajectory of σ_t (middle panel of Figure A.2) consistently features a substantial decline in the elasticity of substitution since the early 1980s.

Thirdly, we have combined the above two scenarios, allowing both for (i) a quality-adjusted labor input, and (ii) Box-Cox labor-augmenting technical progress. Table A.3 summarizes the estimates for this case. The estimated curvature parameter of labor-augmenting technical change λ_l is more convincing now as it is (statistically significantly) below unity, in line with the earlier CES-based results of Klump, McAdam, and Willman (2007). Thus we identify a slight decreasing tendency in the rates of labor-augmenting technical change. More flexibility in the specification of technological progress does not affect our other findings. Signs of all estimated parameters remain consistent across all estimation strategies, whereas σ_t , irrespectively of estimation strategy and the IEES function specification, has been fluctuating until the late 1980s and displayed a substantial decline afterwards (bottom panel of Figure A.2).

Fourthly, we have also tried allowing technical change to be simultaneously labor- and capital-augmenting (left panel of Table A.4). We have then, in three consecutive steps, coupled this extension with inclusion of (i) a quality-adjusted labor input (right panel of Table A.4), (ii) a Box-Cox specification of technological progress (left panel of Table A.5), and (iii) both (right panel of Table A.5). Although the signs of key estimated coefficients are generally preserved in this series of robustness checks, the ensuing results are somewhat less convincing than the previous ones. Crucially, our estimates of the pace of capital-augmenting technical change are consistently negative. This outcome has a bearing on other findings as well: σ_0 is now found to be visibly closer to unity than in the baseline case (though still in the range of gross complementarity); the elasticity of elasticity of substitution ψ is now closer to zero and statistically insignificant in the system approach. The predicted paths of σ_t (Figure A.3) generally agree that σ_t has been declining over the years. They are now less consistent across the four IEES specifications, however, and no longer indicate a qualitative change in the behavior of σ_t after 1980. This is likely because when we agnostically fit a highly nonlinear model featuring both labor- and capital-augmenting technical change to the data, the observed declines in the rate of real investment rate and the capital-output ratio after 1980 (and even more strongly so after the world economic crisis) can potentially be (mis)interpreted as technological regress. Alternatively, however, this result could also be explained by the ongoing *routinization* of production (Acemoglu and Restrepo, 2015) or shifts in the composition of industries (Elsby, Hobijn, and Sahin, 2013; Oberfield and Raval, 2014). We observe that the puzzling result of negative capital-augmenting technical change appears strongest in the system approach which is relatively most demanding of the data (technological progress terms are nested in the $\sigma(k)$ formulas), and is likely further amplified by the fact that our estimation strategy does not allow for markups. Addressing these issues in more detail is left for further research.

7 Conclusion

In the current paper, we have constructed a novel class of normalized Isoelastic Elasticity of Substitution (IEES) production functions and analyzed its properties. Our analytical results are summarized in the following Table 3, expanding upon Table 1 provided in the Introduction. We have also discussed the empirical usefulness of these functions and the ways in which they can be reconciled with factor-augmenting technical change. Our empirical results for the aggregate production function in the post-war US economy imply that the elasticity of substitution σ between capital and labor has been systematically positively related to the capital–labor ratio in effective (technology-adjusted) units, \bar{k} , as well as the capital–output ratio κ . We also observe that σ has been below unity on average, first fluctuating around 0.8–0.9 until the 1980s and then embarking on a secular downward trend.

The scope for further applications of IEES functions is very broad. First, when understood as macroeconomic production functions with capital and labor, they could improve our understanding of the dynamic behavior of factor shares over time as well as their dispersion across countries, regions, sectors, and firms. They could also turn out useful in growth and levels accounting.

Second, when applied to the substitution possibilities across other pairs of inputs, they could be helpful in analyzing the problems of essentiality of exhaustible resources, skill-biased technical change, capital-skill complementarity, aggregation of intermediate goods, aggregation of domestic and imported goods, and preferences over consumption goods with varying degrees of complementarity.

Third, from the point of view of theory, IEES functions may become a useful tool for analyzing long-run growth, poverty traps, medium-run swings, and short-run fluctuations in economic activity. In particular, allowing for an endogenous shift between production factors being gross complements and gross substitutes (as it is possible for IEES(MRS), IEES(k) and IEES(κ) functions) can substantially change long-run predictions of known growth models. Note that physical capital accumulation alone can become an engine of unbounded endogenous growth only if the elasticity of substitution exceeds unity, $\sigma > 1$ (Solow, 1956; Jones and Manuelli, 1990; Palivos and Karagiannis, 2010). Because IEES(MRS), IEES(k) and IEES(κ) functions with $\psi > 0$ imply that $\sigma(k) > 1$ if and only if k is sufficiently large, they may therefore become a useful tool not only in the modeling of endogenous growth, but also poverty traps and multiple equilibria in growth performance. Endogenous shifts between production factors being gross complements and gross substitutes could also enrich our understanding of short- and medium-run responses of factor shares to technological shocks.

Table 3: Summary of Analytical Results

	$\sigma = 1$	$\frac{\sigma}{\sigma_0} = 1$	$\frac{\sigma}{\sigma_0} = \left(\frac{\pi}{1-\pi} \frac{1-\pi_0}{\pi_0}\right)^\psi$	$\frac{\sigma}{\sigma_0} = \left(\frac{\varphi(k)}{\varphi_0}\right)^\psi$	$\frac{\sigma}{\sigma_0} = \left(\frac{k}{k_0}\right)^\psi$
	C-D	CES	IEES $\left(\frac{\pi}{1-\pi}\right)$	IEES(MRS)	IEES(k)
MAIN RESULTS					
Domain	$k \in [0, +\infty)$	$k \in [0, +\infty)$	$\sigma_0 > 1 : k \in [0, +\infty)$ $\sigma_0 < 1, \psi > 0 : k \in [0, k_{max}]$ $\sigma_0 < 1, \psi < 0 : k \in [k_{min}, +\infty)$	$\psi > 0 : k \in [k_{min}, +\infty)$ $\psi < 0 : k \in [0, k_{max}]$	$k \in [0, +\infty)$
$\frac{\pi(k)}{1-\pi(k)}$ (relative factor share)	$\frac{\pi_0}{1-\pi_0}$	$\frac{\pi_0}{1-\pi_0} \left(\frac{k}{k_0}\right)^{\frac{\sigma-1}{\sigma}}$	$\frac{\pi_0}{1-\pi_0} \left(\frac{1}{\sigma_0} + \left(1 - \frac{1}{\sigma_0}\right) \left(\frac{k}{k_0}\right)^\psi\right)^{\frac{1}{1-\psi}}$	$\frac{\pi_0}{1-\pi_0} \frac{k}{k_0} \left(1 + \frac{\psi}{\sigma_0} \ln\left(\frac{k}{k_0}\right)\right)^{-\frac{1}{1-\psi}}$	$\frac{\pi_0}{1-\pi_0} \frac{k}{k_0} e^{-\frac{1}{\psi\sigma_0} \left(1 - \left(\frac{k}{k_0}\right)^{-\psi}\right)}$
MRS, $\varphi(k)$	constant	$\sigma > 1 : \text{increasing}$ $\sigma < 1 : \text{decreasing}$	$\sigma_0 > 1 : \text{increasing}$ $\sigma_0 < 1 : \text{decreasing}$	$\psi > 0 : \cup\text{-shaped}$ $\psi < 0 : \cap\text{-shaped}$	$\psi > 0 : \cup\text{-shaped}$ $\psi < 0 : \cap\text{-shaped}$
$\sigma(k)$	$\varphi_0 \left(\frac{k}{k_0}\right)$	$\varphi_0 \left(\frac{k}{k_0}\right)^{\frac{1}{\sigma}}$	$\varphi_0 \left(\frac{1}{\sigma_0} \left(\frac{k}{k_0}\right)^{-\psi} + \left(1 - \frac{1}{\sigma_0}\right)\right)^{-\frac{1}{1-\psi}}$	$\varphi_0 \left(1 + \frac{\psi}{\sigma_0} \ln\left(\frac{k}{k_0}\right)\right)^{\frac{1}{1-\psi}}$	$\varphi_0 e^{\frac{1}{\psi\sigma_0} \left(1 - \left(\frac{k}{k_0}\right)^{-\psi}\right)}$
(elasticity of substitution)	1	σ_0	$1 + (\sigma_0 - 1) \left(\frac{k}{k_0}\right)^\psi$	$\sigma_0 + \psi \ln\left(\frac{k}{k_0}\right)$	$\sigma_0 \left(\frac{k}{k_0}\right)^\psi$
	constant	constant	$\sigma < 1 (> 1) \Leftrightarrow \sigma_0 < 1 (> 1)$ $\psi(\sigma_0 - 1) > 0 : \text{increasing}$ $\psi(\sigma_0 - 1) < 0 : \text{decreasing}$	$\sigma = 1 \text{ if } \frac{k}{k_0} = e^{-\frac{\sigma_0-1}{\psi}}$ $\psi > 0 : \text{increasing}$ $\psi < 0 : \text{decreasing}$	$\sigma = 1 \text{ if } \frac{k}{k_0} = \sigma_0^{-\frac{1}{\psi}}$ $\psi > 0 : \text{increasing}$ $\psi < 0 : \text{decreasing}$
RESULTS FOR THE CAPITAL DEEPENING REPRESENTATION (WITH $\kappa = k/y$)					
	C-D	CES	IEES (κ)		
$\pi(\kappa)$ (capital share)	π_0	$\pi_0 \left(\frac{\kappa}{\kappa_0}\right)^{\frac{\sigma-1}{\sigma}}$	$\pi_0 \left(\frac{\kappa}{\kappa_0}\right) e^{-\frac{1}{\psi\sigma_0} \left(1 - \left(\frac{\kappa}{\kappa_0}\right)^{-\psi}\right)}$		
$\sigma(\kappa)$ (elasticity of substitution)	1	σ_0	$\sigma_0 \left(\frac{\kappa}{\kappa_0}\right)^\psi$		
	constant	constant	$\sigma > 1 : \text{increasing}$ $\sigma < 1 : \text{decreasing}$	$\psi > 0 : \cup\text{-shaped}$ $\psi < 0 : \cap\text{-shaped}$	
	constant	constant	$\sigma = 1 \text{ if } \frac{\kappa}{\kappa_0} = \sigma_0^{-\frac{1}{\psi}}$ $\psi > 0 : \text{increasing}$ $\psi < 0 : \text{decreasing}$		

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A Appendix

A.1 Data Construction

Our dataset contains long-dated time series for the non-residential business sector in the US economy and covers the time span from 1948Q1 to 2013Q4. The basic data source is BEA NIPA.

Real and nominal output is calculated as follows. First, gross domestic product is reduced by government gross value added and gross output in the housing sector. Then, effects of indirect taxation are subtracted from the data. The data are taken from BEA NIPA tables 1.3.5 (Nominal GDP, GVA), 1.3.6 (Real GDP, GVA), 1.12 (Indirect Taxes less Subsidies). Effects of indirect taxation are eliminated from real output by assuming that its share in real output is the same as in nominal output.

The annual real capital stock in the non-residential business sector is taken from NIPA FAT table 4.2. Because there is no available data on quarterly capital stocks, growth rates of nominal private non-residential fixed investment (BEA NIPA table 1.1.5) are used to interpolate the series. As a result, the obtained quarterly series has the same trend.

BEA does not publish data on the labor input at a quarterly frequency. Instead of interpolating the annual series, we use the BLS series to construct this variable. Our measure of the labor input is a simple sum of the number of employees (FRED code: *USPRIV*) and the self-employed (*LNS12032192*). Since the proposed measure does not take ongoing changes in labor composition into consideration, as a robustness check we use Fernald's (2012) data on quality-adjusted aggregate hours which can be easily converted from annualized growth rates into an index.

To measure factor income shares that are consistent with our definition of output we proceed as follows. The labor share is adjusted by the number of the self-employed in order to deal with the problem of assignment of ambiguous income to either capital or labor (see Mućk, McAdam, and Growiec, 2015, for a wider discussion):

$$\text{Labor share: } 1 - \pi_t = \frac{w_t L_t}{P_t Y_t} = \frac{CE_t}{Output_t} \left(1 + \frac{SE_t}{E_t} \right) \quad (\text{A.1})$$

where CE_t denotes compensation of employees, $Output_t$ is the above described output in nominal terms, SE_t and E_t stand for the number of the self-employed and employees, respectively. The data on SE_t and E_t are consistent with our measure of the labor input. For consistency in terms of the range of economy, CE_t is calculated as the compensation of employees reduced by wages and salaries in the government sector and supplements to wages in this sector.¹⁷ These series are taken from NIPA table 1.12.

¹⁷We assume that the proportion of supplements to wages in the government sector and in the entire economy is the same as the ratio of wages and salaries earned in the government sector to aggregate wages and salaries.

The measurement of the user cost of capital is extremely problematic. We also assume throughout the analysis that the production function has constant returns to scale. Therefore, the capital share is calculated residually:

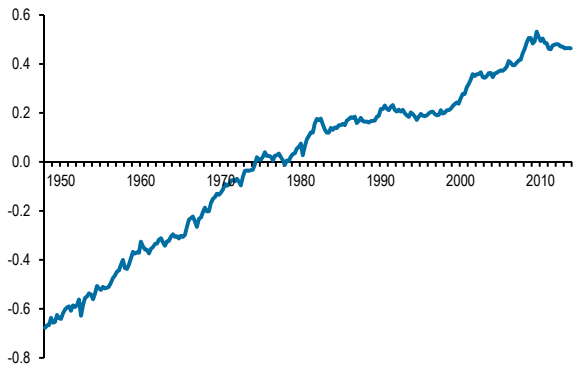
$$\text{Capital share: } \pi_t = \frac{r_t K_t}{P_t Y_t} = 1 - \frac{w_t L_t}{P_t Y_t}. \quad (\text{A.2})$$

As shown by [McAdam and Willman \(2013\)](#), this agnostic approach of measuring the capital share allows to identify correctly the most critical parameters characterizing the supply side of the postwar US economy.

Figure A.1: The US Time Series: 1948Q1:2013Q4

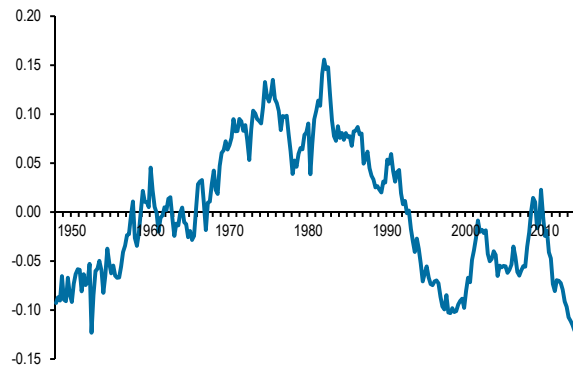
the capital-labor ratio:

$$\ln \left(\frac{k_t}{k_0} \right) = \ln \left(\frac{K_t L_0}{L_t K_0} \right)$$



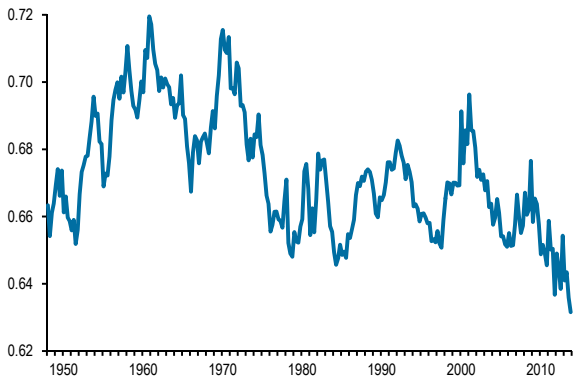
the capital-labor ratio in effective units:

$$\ln \left(\frac{\bar{k}_t}{\bar{k}_0} \right) = \ln \left(\frac{K_t L_0}{L_t K_0} e^{-\gamma_1(t-\bar{t})} \right)$$



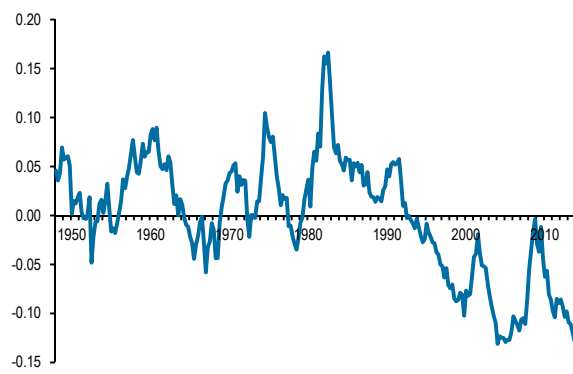
the labor share:

$$1 - \pi_t = \left(\frac{w_t L_t}{P_t Y_t} \right)$$



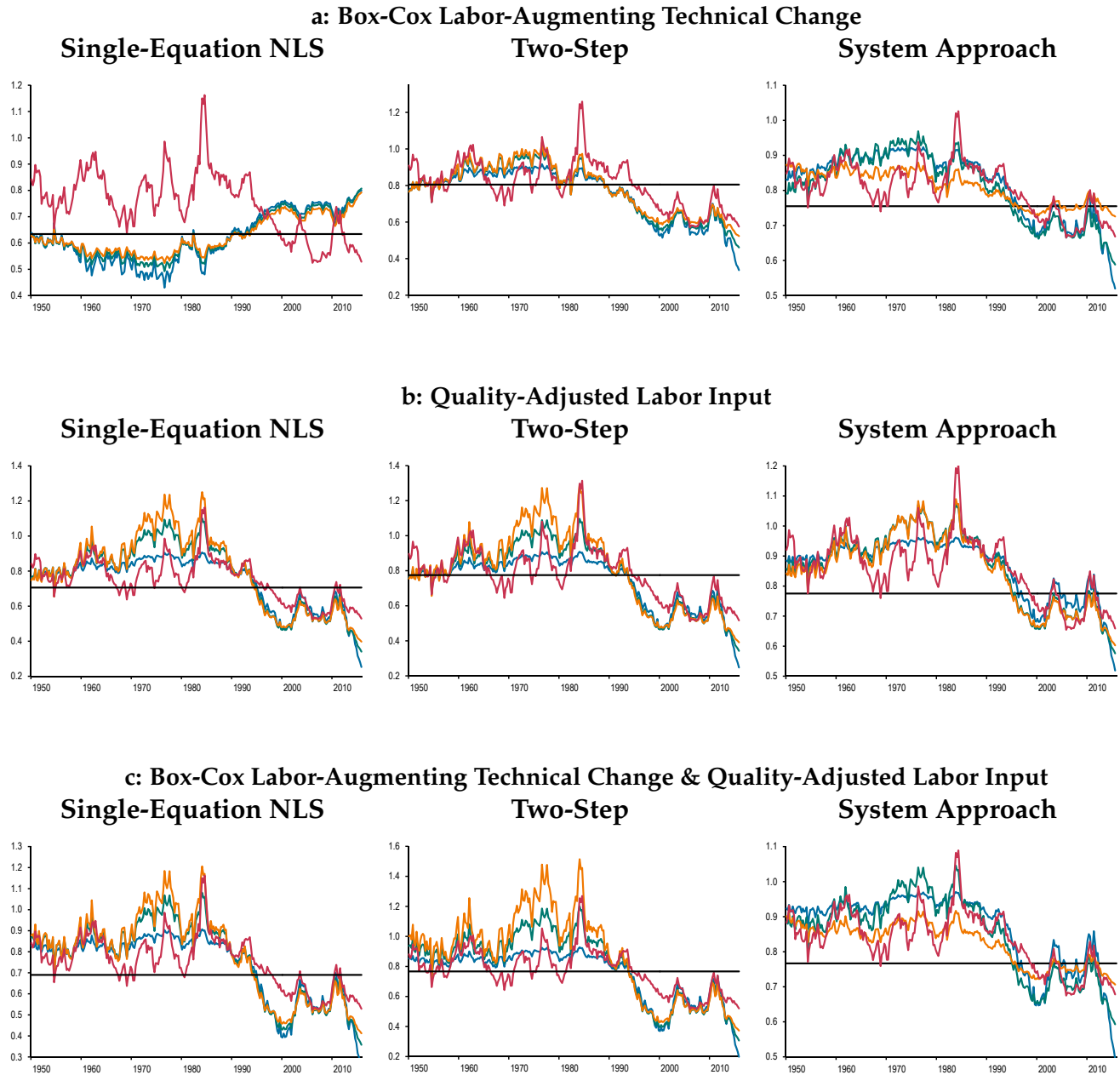
the capital-output ratio:

$$\ln \left(\frac{\kappa_t}{\kappa_0} \right) = \ln \left(\frac{K_t Y_0}{Y_t K_0} \right)$$



A.2 Robustness Checks: Detailed Results

Figure A.2: Implied σ_t : Alternative Specifications



Notes: CES estimates, $\text{IEES}\left(\frac{\pi}{1-\pi}\right)$, $\text{IEES}(MRS)$, $\text{IEES}(k)$ and $\text{IEES}(\kappa)$.

Table A.1: Summary of Estimates of IEES Production Functions - Box-Cox Labor-Augmenting Technical Change

	CES	IEES($\frac{\pi}{1-\pi}$)	IEES(MRS)	IEES(\bar{k})	IEES(κ)
Single-Equation NLS ($\gamma_l = 0.005$ and $\lambda_l = 1.148$)					
π_0	0.325*** (0.002)	0.327*** (0.002)	0.327*** (0.001)	0.326*** (0.001)	0.324*** (0.001)
σ_0	0.634*** (0.019)	0.611*** (0.019)	0.607*** (0.021)	0.609*** (0.023)	0.745*** (0.029)
ψ		3.339** (1.541)	-1.002* (0.567)	-1.318 (0.897)	2.676*** (0.726)
ADF (12), (30), (42), (59)	-3.249***	-3.326***	-3.292***	-3.276***	-2.688***
Two-Step ($\sigma_0 = 0.755$, $\gamma_l = 0.005$ and $\lambda_l = 1.148$)					
π_0		0.323*** (0.001)	0.323*** (0.001)	0.323*** (0.001)	0.324*** (0.001)
ψ		-5.340*** (0.572)	1.719*** (0.338)	2.334*** (0.647)	2.833*** (0.886)
ADF (67)-(70)		-3.182***	-3.108***	-3.054***	-2.694***
System Approach					
π_0	0.330*** (0.001)	0.323*** (0.001)	0.323*** (0.001)	0.328*** (0.001)	0.325*** (0.001)
σ_0	0.755*** (0.001)	0.861*** (0.016)	0.837*** (0.015)	0.802*** (0.015)	0.805*** (0.015)
ψ		-5.913*** (0.673)	1.159*** (0.149)	0.814*** (0.016)	1.457*** (0.244)
ξ	0.988*** (0.004)	0.987*** (0.003)	0.986*** (0.003)	1.010*** (0.004)	0.991*** (0.004)
γ_l	0.005*** (0.0000)	0.005*** (0.000)	0.005*** (0.000)	0.004*** (0.000)	0.005*** (0.000)
λ_l	1.148*** (0.031)	1.188*** (0.026)	1.205*** (0.026)	0.887*** (0.014)	1.165*** (0.030)
ADF (64)	-2.720***	-2.705***	-2.721***	-2.646***	-2.738***
ADF (65)	-3.869***	-3.434***	-3.484***	-2.621***	-3.678***
ADF (66)	-3.206***	-3.219***	-3.220***	-3.049***	-3.214***

Notes: as in Table 2. In case of λ_l , the null hypothesis is that $\lambda_l = 1$ (exponential technical change).

Table A.2: Summary of Estimates of IEES Production Functions - Quality-Adjusted Labor Input

	CES	IEES($\frac{\pi}{1-\pi}$)	IEES(MRS)	IEES(\bar{k})	IEES(κ)
Single-Equation NLS ($\gamma_l = 0.005$)					
π_0	0.327*** (0.001)	0.325*** (0.001)	0.323*** (0.001)	0.321*** (0.001)	0.324*** (0.001)
σ_0	0.706*** (0.024)	0.773*** (0.030)	0.777*** (0.036)	0.768*** (0.035)	0.745*** (0.029)
ψ		-5.790*** (0.944)	2.121*** (0.372)	3.196*** (0.646)	2.676*** (0.726)
ADF (12), (30), (42), (59)	-2.569***	-2.656***	-2.833***	-2.929***	-2.688***
Two-Step ($\sigma_0 = 0.775$ and $\gamma_l = 0.005$)					
π_0		0.324*** (0.001)	0.324*** (0.001)	0.323*** (0.001)	0.324*** (0.001)
ψ		-5.872*** (0.546)	2.102*** (0.308)	3.322*** (0.617)	3.172*** (0.895)
ADF (67)-(70)		-2.968***	-2.977***	-2.975***	-2.708***
System Approach					
π_0	0.332*** (0.001)	0.324*** (0.001)	0.323*** (0.001)	0.323*** (0.001)	0.325*** (0.001)
σ_0	0.775*** (0.001)	0.887*** (0.019)	0.863*** (0.020)	0.846*** (0.019)	0.856*** (0.020)
ψ		-7.061*** (0.905)	1.399*** (0.161)	1.654*** (0.216)	2.045*** (0.327)
ξ	1.001*** (0.002)	1.000*** (0.002)	1.000*** (0.002)	1.000*** (0.002)	1.002*** (0.002)
γ_l	0.005*** (0.000)	0.005*** (0.000)	0.005*** (0.000)	0.005*** (0.000)	0.005*** (0.000)
ADF (64)	-2.681***	-3.140***	-3.133***	-3.100***	-3.002***
ADF (65)	-3.400***	-3.552***	-3.517***	-3.505***	-3.514***
ADF (66)	-3.238***	-3.225***	-3.229***	-3.224***	-3.218***

Notes: as in Table 2.

Table A.3: Summary of Estimates of IEES Production Functions - Quality-Adjusted Labor Input & Box-Cox Labor-Augmenting Technical Change

	CES	IEES($\frac{\pi}{1-\pi}$)	IEES(MRS)	IEES(\bar{k})	IEES(κ)
Single-Equation NLS ($\gamma_l = 0.005$ and $\lambda_l = 0.883$)					
π_0	0.329*** (0.001)	0.325*** (0.001)	0.325*** (0.001)	0.325*** (0.001)	0.324*** (0.001)
σ_0	0.690*** (0.025)	0.744*** (0.034)	0.730*** (0.034)	0.716*** (0.030)	0.745*** (0.029)
ψ		-6.598*** (1.292)	2.290*** (0.545)	3.399*** (0.900)	2.676*** (0.726)
ADF (12), (30), (42), (59)	-2.787***	-3.192***	-3.185***	-3.179***	-2.961***
Two-Step ($\sigma_0 = 0.767$, $\gamma_l = 0.005$ and $\lambda_l = 0.883$)					
π_0		0.325*** (0.001)	0.324*** (0.001)	0.324*** (0.001)	0.324*** (0.001)
ψ		-7.599*** (0.687)	2.831*** (0.439)	4.438*** (0.959)	3.031*** (0.891)
ADF (67)-(70)		-3.191***	-3.184***	-3.161***	-2.994***
System Approach					
π_0	0.333*** (0.001)	0.325*** (0.001)	0.324*** (0.001)	0.328*** (0.001)	0.326*** (0.001)
σ_0	0.767*** (0.001)	0.891*** (0.020)	0.840*** (0.019)	0.806*** (0.018)	0.834*** (0.019)
ψ		-8.534*** (1.274)	1.345*** (0.200)	0.839*** (0.014)	1.609*** (0.344)
ξ	1.012*** (0.003)	1.007*** (0.003)	1.006*** (0.003)	1.025*** (0.004)	1.010*** (0.003)
γ_l	0.005*** (0.000)	0.005*** (0.000)	0.005*** (0.000)	0.005*** (0.000)	0.005*** (0.000)
λ_l	0.883*** (0.019)	0.924*** (0.019)	0.934*** (0.019)	0.813*** (0.011)	0.911*** (0.020)
ADF (64)	-2.693***	-2.698***	-2.717***	-2.700***	-2.739***
ADF (65)	-3.316***	-3.128***	-3.176***	-2.573**	-3.279***
ADF (66)	-3.254***	-3.245***	-3.245***	-3.241***	-3.241***

Notes: as in Table 2. In case of λ_l , the null hypothesis is that $\lambda_l = 1$ (exponential technical change).

Table A.4: Summary of Estimates of IEES Production Functions - Labor- and Capital-Augmenting Technical Change

	RAW LABOR INPUT			QUALITY-ADJUSTED LABOR INPUT						
	CES	IEES($\frac{\pi}{1-\pi}$)	IEES(MRS)	IEES(\bar{k})	IEES(κ)	CES	IEES($\frac{\pi}{1-\pi}$)	IEES(MRS)	IEES(\bar{k})	IEES(κ)
Single-Equation NLS										
	$(\gamma_l = 0.006 \text{ and } \gamma_k = -0.003)$						$(\gamma_l = 0.013 \text{ and } \gamma_k = -0.017)$			
π_0	0.325*** (0.001)	0.325*** (0.001)	0.325*** (0.001)	0.325*** (0.001)	0.324*** (0.001)	0.327*** (0.001)	0.325*** (0.001)	0.324*** (0.001)	0.324*** (0.001)	0.324*** (0.001)
σ_0	0.903*** (0.008)	0.908*** (0.011)	0.906*** (0.009)	0.906*** (0.008)	0.745*** (0.029)	0.977*** (0.003)	0.979*** (0.003)	0.978*** (0.002)	0.978*** (0.002)	0.978*** (0.002)
ψ	-0.772* (0.455)	0.088* (0.046)	0.100** (0.051)	0.100** (0.051)	2.676*** (0.726)	-0.222** (0.104)	0.007*** (0.002)	0.007*** (0.002)	0.007*** (0.002)	0.010*** (0.004)
ADF	-3.148***	-3.267***	-3.248***	-3.245***	-2.998***	-3.209***	-3.224***	-3.192***	-3.191***	-3.182***
Two-Step										
	$(\gamma_l = 0.006, \gamma_k = -0.003 \text{ and } \sigma_0 = 0.883)$						$(\gamma_l = 0.013, \gamma_k = -0.017 \text{ and } \sigma_0 = 0.973)$			
π_0	0.325*** (0.001)	0.325*** (0.001)	0.325*** (0.001)	0.325*** (0.001)	0.323*** (0.001)	0.327*** (0.001)	0.325*** (0.001)	0.324*** (0.001)	0.324*** (0.001)	0.324*** (0.001)
ψ	-0.700** (0.340)	0.087** (0.042)	0.100** (0.047)	0.100** (0.047)	5.636*** (0.917)	-0.198** (0.078)	0.007*** (0.002)	0.007*** (0.002)	0.007*** (0.002)	0.010*** (0.003)
ADF (67)-(70)	-3.277***	-3.246***	-3.240***	-3.223***	-3.223***	-3.193***	-3.245***	-3.193***	-3.191***	-3.177***
System Approach										
π_0	0.326*** (0.001)	0.325*** (0.001)	0.325*** (0.001)	0.325*** (0.001)	0.328*** (0.001)	0.327*** (0.001)	0.326*** (0.001)	0.325*** (0.001)	0.325*** (0.001)	0.325*** (0.001)
σ_0	0.883*** (0.001)	0.919*** (0.024)	0.922*** (0.024)	0.922*** (0.024)	0.909*** (0.016)	0.973*** (0.002)	0.972*** (0.008)	0.972*** (0.008)	0.972*** (0.008)	0.970*** (0.009)
ψ	-0.423* (0.218)	0.037 (0.026)	0.041 (0.030)	0.041 (0.030)	-0.132 (0.139)	-0.080 (0.049)	0.003 (0.002)	0.003 (0.002)	0.003 (0.002)	0.004 (0.002)
ξ	1.004*** (0.002)	1.003*** (0.002)	1.003*** (0.002)	1.003*** (0.002)	1.002*** (0.002)	1.013*** (0.003)	1.008*** (0.003)	1.008*** (0.003)	1.008*** (0.003)	1.008*** (0.003)
γ_l	0.006*** (0.000)	0.006*** (0.001)	0.006*** (0.001)	0.006*** (0.001)	0.006*** (0.000)	0.013*** (0.001)	0.012*** (0.002)	0.011*** (0.002)	0.011*** (0.002)	0.011*** (0.002)
γ_k	-0.003*** (0.000)	-0.004*** (0.001)	-0.004*** (0.002)	-0.004*** (0.002)	-0.004*** (0.001)	-0.017*** (0.001)	-0.013*** (0.004)	-0.013*** (0.004)	-0.013*** (0.004)	-0.013*** (0.004)
ADF (64)	-3.476***	-3.484***	-3.478***	-3.477***	-3.455***	-3.615***	-3.556***	-3.557***	-3.557***	-3.557***
ADF (65)	-3.636***	-3.786***	-3.775***	-3.775***	-3.723***	-3.681***	-3.643***	-3.647***	-3.650***	-3.651***
ADF (66)	-2.913***	-2.915***	-2.905***	-2.904***	-2.940***	-2.503**	-2.650***	-2.646***	-2.646***	-2.651***

Notes: as in Table 2.

Table A.5: Summary of Estimates of IIES Production Functions - Labor- and Capital-Augmenting Technical Change (Box-Cox)

	RAW LABOR INPUT			QUALITY-ADJUSTED LABOR INPUT		
	CES	$IIES(\frac{\pi}{1-\pi})$	$IIES(MRS)$	$IIES(\frac{\pi}{1-\pi})$	$IIES(MRS)$	$IIES(\bar{k})$
Single-Equation NLS						
π_0	0.326*** (0.001)	0.324*** (0.001)	0.325*** (0.001)	0.324*** (0.001)	0.325*** (0.001)	0.325*** (0.001)
σ_0	0.964*** (0.008)	0.930*** (0.009)	0.905*** (0.006)	0.939*** (0.008)	0.917*** (0.006)	0.918*** (0.006)
ψ	-1.731*** (0.383)	-1.731*** (0.383)	0.083*** (0.011)	-1.510*** (0.339)	0.061*** (0.008)	0.061*** (0.008)
ADF	-3.062***	-3.246***	-3.216***	-3.204***	-3.159***	-3.162***
Two-Step						
π_0	0.325*** (0.001)	0.325*** (0.001)	0.325*** (0.001)	0.325*** (0.001)	0.325*** (0.001)	0.326*** (0.001)
ψ	-1.350*** (0.221)	0.082*** (0.005)	0.083*** (0.005)	-1.192*** (0.201)	0.061*** (0.004)	0.061*** (0.004)
ADF (67)-(70)	-3.376***	-3.206***	-3.219***	-3.327***	-3.150***	-3.162***
System Approach						
π_0	0.323*** (0.001)	0.327*** (0.001)	0.326*** (0.001)	0.325*** (0.001)	0.328*** (0.001)	0.324*** (0.001)
σ_0	0.895*** (0.001)	0.810*** (0.002)	0.897*** (0.016)	0.911*** (0.001)	0.928*** (0.014)	0.931*** (0.016)
ψ	-0.493*** (0.109)	-0.493*** (0.109)	-0.024 (0.022)	-0.201** (0.095)	0.036** (0.014)	0.053** (0.027)
ξ	0.993*** (0.004)	0.992*** (0.004)	0.990*** (0.004)	1.014*** (0.003)	1.021*** (0.003)	1.019*** (0.003)
γ_1	0.006*** (0.000)	0.005*** (0.000)	0.006*** (0.001)	0.007*** (0.000)	0.008*** (0.001)	0.008*** (0.001)
λ_1	1.219*** (0.068)	1.176*** (0.050)	1.250*** (0.124)	0.974*** (0.053)	0.623*** (0.041)	0.654*** (0.043)
γ_k	-0.004*** (0.000)	-0.001*** (0.000)	-0.003*** (0.001)	-0.005*** (0.000)	-0.006*** (0.001)	-0.006*** (0.001)
λ_k	1.358*** (0.248)	1.442*** (0.454)	1.478*** (0.491)	1.167*** (0.193)	0.362*** (0.066)	0.405*** (0.068)
ADF (64)	-3.565***	-3.416***	-3.494***	-3.590***	-3.564***	-3.546***
ADF (65)	-3.919***	-4.082***	-3.883***	-3.759***	-3.755***	-3.734***
ADF (64)	-2.579**	-2.720***	-2.736***	-2.472**	-2.520**	-2.551**

Notes: as in Table 2. In cases of λ_1 and λ_k , the null hypotheses are that $\lambda_1 = 1$ and $\lambda_k = 1$ (exponential technical change).

Figure A.3: Implied σ_t : Alternative Specifications

