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problem in DSGE models

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# A solution to the global identification problem in DSGE models\*

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## Abstract

We develop an analytical framework to study global identification in structural models with forward-looking expectations. Our identification condition combines the similarity transformation linking the observationally equivalent state space systems with the constraints imposed on them by the model parameters. The key step of solving the identification problem then reduces to finding all roots of a system of polynomial equations. We show how it can be done using the concept of a Gröbner basis and recently developed algorithms to compute it analytically. In contrast to papers relying on numerical search, our approach can prove whether a model is identified or not at a given parameter point, explicitly delivering the complete set of observationally equivalent parameter vectors. We present the solution to the global identification problem for several popular DSGE models.

*JEL:* C10, C51, C65, E32

*Keywords:* global identification, state space systems, DSGE models, Gröbner basis

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# 1 Introduction

Parameter identification is one of the primary concerns of structural macroeconomic modeling. In the context of traditional simultaneous equations systems, the essence of the problem and its treatment has already been formalized in the 1940s, mainly by various authors connected to the Cowles Commission for Research in Economics (see e.g. Koopmans, 1949). In recent decades, this class of purely backward-looking models has been gradually replaced, in both academic circles and policy making institutions, by the so-called dynamic stochastic general equilibrium (DSGE) models. In these mathematical constructs, the dynamics is driven by unobserved stochastic processes and crucially depends on agents' expectations, typically assumed rational. The key difficulty with this class of models in the context of identification is that while their solution has a state-space representation, for which the global identification problem is fairly well understood, the coefficients defining this solution are only implicit rather than analytical functions of the original model parameters. As a result, a new approach to identification became necessary.

Early contributions highlighting the identification problem in simple DSGE models include Beyer and Farmer (2007), Fukac et al. (2007), Canova and Sala (2009) and Cochrane (2011). A more formal analysis soon followed, focusing first on local identification issues, and resulting in the rank conditions on an appropriately defined Jacobian matrix (Iskrev, 2010; Komunjer and Ng, 2011) or spectral density matrix (Qu and Tkachenko, 2012). Important progress has also been made towards resolving the problem of global identification. Qu and Tkachenko (2017) present a numerical routine that searches for observationally equivalent parameters by minimizing the Kullback-Leibler distance in the frequency domain. Kocięcki and Kolasa (2018) develop an alternative algorithm that relies on the conditions linking observationally equivalent state space representations from Komunjer and Ng (2011), thus avoiding the need to solve the model for each candidate parameter.<sup>1</sup>

However, while these two existing approaches to global identification problem are very useful tools for detecting possible identification failure, they have one important limitation in that they cannot strictly prove that a given model is globally identified. If a numerical search routine fails to find an observationally equivalent vector of parameters to the one at which one checks identification, this does not necessarily mean that such a vector does not exist. It might be that the algorithm simply neglected some support of the (multi-dimensional) parameter space, where observationally equivalent points are situated. Therefore, the lack of a solution to the problem of global identification in DSGE models should be considered a serious methodological gap.

Against this backdrop, this paper develops an analytical framework to study global identification in dynamic linear systems with rational expectations. The framework is comprehensive in that it encompasses both determinate models, in which the rational expectations solution is unique, as well as indeterminate ones, where the dynamics may be additionally driven by sunspot shocks. The essence of our approach consists of two insights. The first one establishes a formal identification condition that reduces checking identification of the model's parameters (or their appropriately

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<sup>1</sup>While all of this literature deals with linearized DSGE models, there have been some attempts to study local identification of their higher-order approximations, see e.g. Mutschler (2015).

defined analytical functions) to finding all roots of a system of polynomial equations. This condition is derived by linking the observationally equivalent state space systems with the inherent constraints imposed on them by the deep parameters of the underlying structural model. The second insight relies on applying the concept of the Gröbner basis to analytically solve this system of polynomial equations. In short, and postponing the details for later, this last step boils down to transforming the original system of polynomials into an equivalent triangular system, which is done in a way resembling Gaussian elimination in linear algebra.

The key advantage of our framework is that it explicitly derives and directly checks the global identification conditions at a given point in the parameter space. This gives the formal proof of global identification or lack thereof at this point as the calculation of the Gröbner basis is exact in principle. Our explicit approach is hence an advantage over the two existing methods to check global identification in DSGE models (Kocięcki and Kolasa, 2018; Qu and Tkachenko, 2017), both of which rely on searching numerically over the parameter space, and hence cannot formally prove that the model is identified. Another useful feature of our framework is that it generalizes and unifies the conditions linking observationally equivalent state space representations, which in the previous identification literature were derived separately for singular and non-singular cases. While designed to solve the global identification problem, our framework offers also some additional insights over the existing and well-established approaches to handle its local variant. They can determine which parameters need to be fixed to obtain identification but they do not explicitly link their observationally equivalent values or can approximate such links only numerically as done by Qu and Tkachenko (2012). In contrast, a Gröbner basis applied to our identification condition explicitly produces the complete set of parameter vectors that are observationally equivalent to the one at which one checks identification.

Gröbner basis methods are a fast developing field in computational algebraic geometry. Despite their great potential, they are still very rarely used in economics, with only few exceptions. Kubler and Schmedders (2010a) and Kubler and Schmedders (2010b) successfully apply these methods to determine the exact number of equilibria in several economic models and to calculate them analytically. Foerster et al. (2016) apply Gröbner bases to obtain higher-order approximations to the solutions of Markov-switching DSGE models. Datta (2010) exploits this concept to find Nash equilibria in games. Our contribution is to apply it to identification analysis of linearized DSGE models. It needs to be stressed that the Gröbner basis-driven identification analysis of linear state-space systems has long tradition in fields other than economics. Among the earliest examples in the engineering literature, one can mention Walter and Lecourtier (1982) and Lecourtier and Raksanyi (1985). In general, one strand in this literature, started with Ljung and Glad (1994), combines differential algebra with Gröbner basis techniques, see e.g. Pia Saccomani et al. (2001), the other one draws more heavily on algebraic geometry, see e.g. Meshkat et al. (2009), Ovchinnikov et al. (2021). These approaches have become useful in biological, epidemiological, medical and chemical applications to the extent that they culminated with a number of dedicated algorithms implemented in many computer algebra languages – see e.g. DAISY in REDUCE software (Bellu et al., 2007),

SIAN in MAPLE software (Hong et al., 2020), or COMBOS with web-based application (Meshkat et al., 2014). These algorithms are confined to a differential equations setup and work under clearly different (noise-free) notion of structural identification. They also make an implicit assumption that the model’s coefficients are analytically known, which is a rare exception for the state space representation of a DSGE model. In the statistics literature, the work that is closest to ours is the recent paper by Duan et al. (2020), who vividly promote Gröbner basis methods to solve global identification problems in many popular statistical (and econometric) models, e.g. binary choice models with misclassification, whose local identification was studied e.g. by Hausman et al. (1998).

Despite the existence of analytical algorithms that are proved to succeed after a finite number of iterations, computing a Gröbner basis for large systems of equations can be quite time and memory consuming in practice. However, there are several features of our application that help alleviate this curse of dimensionality. The key one is that, for a typical DSGE model, the system of polynomials generated by our identification condition is of limited degree and very sparse. In fact, we show that our identification analysis can be applied not only to small-scale DSGE models, but also to their richer versions represented e.g. by a variant of the Smets and Wouters (2007) setup, in which we remove the flexible price block. We show that observational equivalence can be ruled out by fixing only two structural parameters, which turn out to be those that are locally unidentified in an obvious way. We also apply our framework to study identification in several variants of open economy DSGE models, including those featuring a line of promising extensions suggested by the recent literature. Strikingly, we can prove that all of them are globally identified, at least for a standard selection of observable variables. Overall, our findings indicate that ensuring local identification in medium-sized DSGE models will often also make them globally identified.

The rest of this paper proceeds as follows. Section 2 presents the setup and establishes notation for a typical dynamic linear system with rational expectations and its state-space representation. Section 3 derives the conditions linking observationally equivalent state-space representations. Section 4 combines these links with the original (structural) form of the model to establish the formal global identification conditions. Section 5 offers a brief introduction to the concept of Gröbner basis and describes its application to checking the identification condition. Several illustrative examples, including popular DSGE models from the literature and their extensions, are presented in Section 6. Section 7 concludes and discusses some possible further research directions. All proofs and more involved analytical details are relegated to the Appendix.

## 2 Structural model

Let us cast a linearized DSGE model in the following general form

$$\Gamma_0(\theta) \begin{bmatrix} s_t \\ p_t \end{bmatrix} = \Gamma_1(\theta) \mathbb{E}_t \begin{bmatrix} s_{t+1} \\ p_{t+1} \end{bmatrix} + \Gamma_2(\theta) s_{t-1} + \Gamma_3(\theta) \varepsilon_t \quad (1)$$

where  $s_t$  is an  $n \times 1$  vector of states,  $p_t$  is a  $q \times 1$  vector of other endogenous (policy) variables (both expressed in deviation from their steady-state values), and  $\varepsilon_t \sim i.i.d. N(0, \Sigma(\theta))$  is a  $k \times 1$  vector of shocks, which can include both innovations to structural (fundamental) disturbances, sunspot shocks and measurement errors, the last two entering with zero loadings. Matrices  $\Gamma_0(\theta)$ ,  $\Gamma_1(\theta)$ ,  $\Gamma_2(\theta)$ ,  $\Gamma_3(\theta)$  and symmetric positive definite  $k \times k$  matrix  $\Sigma(\theta)$  are explicit functions of deep model parameters collected in an  $m \times 1$  vector  $\theta \in \Theta \subseteq \mathbb{R}^m$ .

A dynamic solution to (1) can be written as

$$s_t = A(\theta)s_{t-1} + B(\theta)\varepsilon_t \quad (2)$$

$$p_t = F(\theta)s_{t-1} + G(\theta)\varepsilon_t \quad (3)$$

where  $A(\theta)$  is an  $n \times n$  matrix,  $B(\theta)$  is an  $n \times k$  matrix,  $F(\theta)$  is a  $q \times n$  matrix and  $G(\theta)$  is a  $q \times k$  matrix, all of which implicitly depend on deep model parameters  $\theta$ . This is always the case if the non-explosive equilibrium is unique. Under indeterminacy, the solution has still the form given by equations (2)-(3) as long as one allows for a sufficient number of sunspot shocks in  $\varepsilon_t$ , see Lubik and Schorfheide (2003). This becomes even more straightforward if, in the case of indeterminacy, one equivalently transforms the model as suggested by Farmer et al. (2015), i.e. redefines a sufficient number of errors in expectations as fundamentals.

Suppose the measurement equations relates the model variables to the data as follows

$$y_t = M(\theta) + H(\theta) \begin{bmatrix} s_t \\ p_t \end{bmatrix} + J(\theta)\varepsilon_t \quad (4)$$

where  $y_t$  is an  $r \times 1$  vector of observable variables,  $M(\theta)$  is an  $r \times 1$  vector,  $H(\theta)$  is an  $r \times (n + q)$  matrix and  $J(\theta)$  is an  $r \times k$  matrix, all of which explicitly depend on  $\theta$ . Decomposing  $H(\theta)$  into blocks corresponding to the state and policy variables  $H(\theta) = [ H^s(\theta) \quad H^p(\theta) ]$  and using equations (2) and (3) allows us to rewrite measurement equation (4) as

$$y_t = M(\theta) + C(\theta)s_{t-1} + D(\theta)\varepsilon_t \quad (5)$$

where an  $r \times n$  matrix  $C(\theta)$  and an  $r \times k$  matrix  $D(\theta)$  are defined as

$$C(\theta) = H^s(\theta)A(\theta) + H^p(\theta)F(\theta) \quad (6)$$

$$D(\theta) = H^s(\theta)B(\theta) + H^p(\theta)G(\theta) + J(\theta) \quad (7)$$

Consequently, the law of motion for observable variables  $y_t$  has a state space form, given by transition equation (2) and measurement equation (5). For future reference, and following Fernández-Villaverde et al. (2007), such a representation will be called the ABCD-representation.

### 3 Observational equivalence of state-space representations

One of the key insights from Komunjer and Ng (2011) is that the ABCD-representation of a DSGE model is not identified, and hence its elements cannot be treated as reduced-form parameters. In this section we generalize their results by developing a set of conditions linking the observationally equivalent ABCD-representations that encompass both singular and non-singular cases.<sup>2</sup> From now on, to save on notation, let us denote any matrix  $X(\theta)$  that depends on  $\theta$  simply as  $X$ . Similarly, when referring to this matrix evaluated at an alternative parameter vector  $\bar{\theta}$ , we will write in short  $\bar{X}$ .

#### 3.1 Theoretical setup

To proceed, we need two assumptions to get our most general identification result for the ABCD-representation of a DSGE model. The first one concerns stability of the model solution.

**Assumption 1.** (*Stability*) For every  $\theta \in \Theta$  and for any  $z \in \mathbb{C}$  (a set of complex numbers)  $\det(z\mathbf{I}_n - A) = 0$  implies  $|z| < 1$ .

The purpose of Assumption 1 is to restrict the analysis to stationary models. As a consequence, we can define the steady-state value  $P = E(s_t s_t')$ , which is a unique solution to the Lyapunov equation  $P = APA' + B\Sigma B'$  implied by equation (2). Bearing in mind measurement equation (5), the autocovariance sequence  $\Lambda_l = E(\tilde{y}_t \tilde{y}_{t-l}')$ , where  $\tilde{y}_t = y_t - M$ , is readily seen as  $\Lambda_0 = CPC' + D\Sigma D'$  and  $\Lambda_l = CA^{l-1}N$ , for  $l > 0$ , where  $N = APC' + B\Sigma D'$ . Needless to say, we have  $\Lambda_{-l} = \Lambda_l'$ .

To state the second assumption, let us define  $\mathcal{O} = [C':A'C':A^2C':\dots:A^{n-1}C']'$  and  $\mathcal{C} = [N:AN:A^2N:\dots:A^{n-1}N]$ .

**Assumption 2.** (*Stochastic minimality*) For every  $\theta \in \Theta$ , matrices  $\mathcal{O}$  and  $\mathcal{C}$  have, respectively, full column and full row rank, i.e.  $\text{rank}(\mathcal{O}) = \text{rank}(\mathcal{C}) = n$ .

Assumption 2 (under the name that we use) is well known in the linear system literature, see e.g. Lindquist and Picci (1996). It is exactly the same as in e.g. Komunjer and Zhu (2020), who term it as autocovariance minimality. Its main purpose is to confine the analysis only to those ABCD-representations (consistent with given autocovariance sequences) in which the dimension of the state vector is as small as possible. To this end, Assumption 2 ensures that the underlying infinite block Hankel matrix made up of the covariances sequence has the (maximal) rank  $n$  for all  $\theta \in \Theta$  (see Appendix A.2).

Assumption 2 differs from the assumptions made by Komunjer and Ng (2011) in how matrix  $\mathcal{C}$  is defined. In their framework,  $N$  is replaced either by  $B$  (Assumption 5-S, applicable to the singular case) or the steady-state Kalman gain associated with the innovations representation of the original state-space system (Assumption 5-NS, for the non-singular case). Moreover, Komunjer

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<sup>2</sup>Non-singular models are the cases in which there are more shocks than observables ( $k > r$ ) or when the system is square ( $k = r$ ) but non-invertible. See the table on page 2010 in Komunjer and Ng (2011).

and Ng (2011) additionally impose left-invertibility of the transfer function (Assumption 4-S for the singular case)<sup>3</sup> or full row rank of matrix  $D$  (Assumption 4-NS for the non-singular case). In our most general form of the identification condition, we do not need any of these additional assumptions. We also do not have to distinguish between singular and non-singular models, which spares us reformulation of the original problem into its innovations representation in the latter case. In this sense, our framework can be seen both as unification and some generalization (as it relies on weaker conditions) of that developed by Komunjer and Ng (2011).

Even though well established in the linear system literature, one may question the practical aspect of Assumption 2 since it is impossible to check its validity for all  $\theta \in \Theta$ . However, in Appendix A.1 we show that if Assumption 2 is valid for some  $\theta$  at which we check identification, then in fact it holds for almost all  $\theta \in \Theta$ . This allows us to safely proceed with our analysis, with the understanding that the underlying deep parameter space  $\Theta$  excludes those  $\theta$ 's that violate the assumption, which however form the nowhere dense subset of measure zero. In fact, as we discuss at the end of this section using a simple  $MA(1)$  model example, the excluded parameter values can just correspond to particular degenerate cases, and hence are not the relevant candidates for observational equivalence to those  $\theta$ 's for which Assumption 2 holds.

As implied by our model formulation, we deal with a stationary linear Gaussian environment. This allows us to define observational equivalence in a standard way (see, e.g., definition 1 in Rothenberg, 1971), by using only the first and second moments as they fully characterize linear Gaussian models. More formally, let us define the  $z$ -spectrum of the ABCD-representation as  $\Phi(z) = H(z)\Sigma H'(z^{-1})$ , where  $H(z) = D + C(zI_n - A)^{-1}B$  is the transfer function (hence  $z^{-1}$  corresponds to its backward shift) and which is well defined for all  $z \in \mathbb{C}$  in some open annulus containing the unit circle, i.e.  $|z| = 1$ , so that the spectral density of the ABCD-representation is also well defined. In fact, such an annulus exists by our Assumption 1, see e.g. Komunjer and Zhu (2020), Lindquist and Picci (1996), Lindquist and Picci (2015), pp. 196-199, and our Appendix A.2. Then we have the following definition

**Definition 1.**  $\theta$  and  $\bar{\theta}$  are observationally equivalent (written as  $\theta \equiv \bar{\theta}$ ) if  $\bar{M} = M$  and  $\bar{\Phi}(z) = \Phi(z)$  for all  $z \in \mathbb{C}$  in an open annulus containing the unit circle.

What Definition 1 conveys is that two deep parameters sets are observationally equivalent if they result in the same means and autocovariance sequence of observable variables, so that we cannot distinguish between these two alternative parametrizations using the first and second moments of the data.<sup>4</sup> We are now ready to state the key theorem.

**Theorem 1.** *Let Assumptions 1 and 2 hold. Then  $\theta \equiv \bar{\theta}$  if and only if 1)  $\bar{A} = TAT^{-1}$ , 2)  $\bar{C} = CT^{-1}$ , 3)  $AQA' - Q = T^{-1}\bar{B}\bar{\Sigma}\bar{B}'T^{-1} - B\Sigma B'$ , 4)  $CQC' = \bar{D}\bar{\Sigma}\bar{D}' - D\Sigma D'$ , 5)  $AQC' = T^{-1}\bar{B}\bar{\Sigma}\bar{D}' - B\Sigma D'$ ,*

<sup>3</sup>Under left-invertibility, finding all observationally equivalent ABCD-representations closely resembles the so-called deterministic realization problem, hence standard assumptions concerning observability and controllability are sufficient.

<sup>4</sup>Observationally equivalent parameter sets as defined by Definition 1 can still generate different moments of unobservable variables and imply different impulse response functions for any variable in the model.



6)  $\bar{M} = M$ , for some nonsingular matrix  $T$  and symmetric matrix  $Q$ . In addition, if  $\theta \equiv \bar{\theta}$  then both  $T$  and  $Q$  are unique.

This theorem is an adapted version of Corollary 4.5 in Glover (1973), which probably belongs to “folk wisdom” among specialists in linear system theory.<sup>5</sup> From the perspective of identification in DSGE models, Theorem 1 generalizes and unifies the key propositions 1-S and 1-NS in Komunjer and Ng (2011), who consider separately the singular and non-singular cases, for which they need to assume left-invertibility of the transfer function in the former case and the full row rank of  $D$  in the latter case. Most importantly, the general form of the theorem allows us to treat the case  $r < k$ , which arises naturally under indeterminacy as full characterization of the model solutions requires adding sunspot shocks (Lubik and Schorfheide, 2003).

It may be useful to know under what further conditions our Theorem 1 nests the conclusions of Propositions 1-S and 1-NS in Komunjer and Ng (2011) for the singular and non-singular case, respectively. Starting with the latter, let us define the Riccati equation (in symmetric matrix  $\Pi$ )

$$\Pi = A\Pi A' + B\Sigma B' - K\Sigma_a K' \quad (8)$$

where  $\Sigma_a = C\Pi C' + D\Sigma D'$  and  $K = (A\Pi C' + B\Sigma D')\Sigma_a^{-1}$  (in what follows we assume that  $\Sigma_a$  is positive definite). Let us also formulate the following assumption.

**Assumption 3.** *For every  $\theta \in \Theta$ , the Riccati equation (8) possesses a unique, positive semidefinite solution.*

Clearly, Assumption 3 is a high level assumption. However, as we show in Appendix A.3, checking whether it holds is quite easy. Then we have the following proposition.

**Proposition 1.** *Let Assumptions 1, 2 and 3 hold. Then  $\theta \equiv \bar{\theta}$  if and only if 1)  $\bar{A} = TAT^{-1}$ , 2)  $\bar{C} = CT^{-1}$ , 3)  $\bar{K} = TK$ , 4)  $\bar{\Sigma}_a = \Sigma_a$ , 5)  $\bar{M} = M$ , for some nonsingular matrix  $T$ . In addition, if  $\theta \equiv \bar{\theta}$  then  $T$  is unique.*

The conclusions of this proposition, which we prove in Appendix A.4, are exactly as in Proposition 1-NS in Komunjer and Ng (2011). Obviously, from an operational point of view, they should be read together with the definition of Riccati equation (8), which links  $K$  and  $\Sigma_a$  ( $\bar{K}$  and  $\bar{\Sigma}_a$ ) to the ABCD-representation via matrix  $\Pi$  ( $\bar{\Pi}$ ).

Let us now move to the case, in which the number of observable variables is equal to the number of shocks, i.e.  $r = k$ . This is by far the most relevant case in the DSGE literature, which uses likelihood-based methods to estimate the model parameters. To nest the square case in our framework, we need the following assumption.

**Assumption 4.** *For every  $\theta \in \Theta$ ,  $D$  is nonsingular.*

Needless to say, as in the case of Assumption 2, if Assumption 4 holds for one  $\theta \in \Theta$ , then it applies for almost all  $\theta$ 's. Then we have the next proposition.

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<sup>5</sup>Since Glover (1973) contains only the proof of a continuous time version of Corollary 4.5, we prove Theorem 1 in Appendix A.2.

**Proposition 2.** *Let Assumptions 1-4 hold. Then  $\theta \equiv \bar{\theta}$  if and only if 1)  $\bar{A} = TAT^{-1}$ , 2)  $\bar{B} = TBU$ , 3)  $\bar{C} = CT^{-1}$ , 4)  $\bar{D} = DU$ , 5)  $\bar{\Sigma} = U^{-1}\Sigma U'^{-1}$ , 6)  $\bar{M} = M$ , for some nonsingular matrix  $T$  and nonsingular matrix  $U$ . In addition, if  $\theta \equiv \bar{\theta}$  then both  $T$  and  $U$  are unique.*

The conclusions of this proposition are the same as in Proposition 1-S in Komunjer and Ng (2011). From the perspective of deep parameter identification that we describe in the following section, the conditions in Proposition 2 are a bit more convenient to handle than those stated in Theorem 1, so we recommend using the former whenever Assumptions 3 and 4 are satisfied.

### 3.2 Simple time series examples

To illustrate the meaning and interaction between our all assumptions, theorem and propositions, we consider two simple yet widely used time series models:  $MA(1)$  and  $ARMA(1, 1)$ . Let us start with the former, i.e.  $y_t = \phi\varepsilon_{t-1} + \varepsilon_t$ , where  $\varepsilon_t \sim N(0, \sigma^2)$ . This model is nested in the ABCD-representation by putting  $A = 0$ ,  $B = \phi$ ,  $M = 0$ ,  $C = 1$ ,  $D = 1$ ,  $\Sigma = \sigma^2$ . Since  $A = 0$ , our model is stable (i.e., Assumption 1 holds). Moreover,  $rank(\mathcal{O}) = rank(C) = 1$  for all  $\theta$ , but  $rank(C) = rank(N) = rank(\phi\sigma^2) = 1$  only for  $\phi \neq 0$ . Evidently, Assumption 2 simply excludes the white noise model (i.e.  $\phi = 0$ ) from the considerations, i.e. a point at which  $\mathcal{C}$  drops rank. Note that this exclusion is not restrictive as white noise cannot be observationally equivalent to any non-degenerate  $MA(1)$  model.

In the case  $\phi \neq 0$ , we can safely apply Theorem 1. Since  $C$  is restricted to 1, we immediately have  $T = 1$ . Then, from 3), 4) and 5) in this theorem, we get  $Q = \phi^2\sigma^2 - \bar{\phi}^2\bar{\sigma}^2$ ,  $Q = \bar{\sigma}^2 - \sigma^2$  and  $\phi\sigma^2 = \bar{\phi}\bar{\sigma}^2$ , respectively. Solving these three equations in three unknowns  $Q, \bar{\phi}, \bar{\sigma}^2$  gives us exactly two solutions  $(Q, \bar{\phi}, \bar{\sigma}^2) = (0, \phi, \sigma^2)$  and  $(Q, \bar{\phi}, \bar{\sigma}^2) = (\sigma^2(\phi^2 - 1), \frac{1}{\phi}, \phi^2\sigma^2)$  for  $\phi \neq \pm 1$ , and one solution  $(Q, \bar{\phi}, \bar{\sigma}^2) = (0, \phi, \sigma^2)$  for  $\phi = \pm 1$ . Hence, the model is not globally identified at  $\phi \neq \pm 1$ .

Let us now demonstrate how our identification analysis of the  $MA(1)$  model works in more specialized cases addressed by Propositions 1 and 2, which require fulfilling Assumption 3.<sup>6</sup> In our  $MA(1)$  example, the Riccati equation (8) possesses two solutions:  $\Pi = 0$  and  $\Pi = \sigma^2(\phi^2 - 1)$ . For Assumption 3 to hold, we hence need to restrict  $|\phi| < 1$  as then the only positive semidefinite solution to the Riccati equation is  $\Pi = 0$ . As a matter of fact, in Appendix A.3 we show that Assumption 3 holds if and only if  $\Psi = A - BD^{-1}C = -\phi$  is stable, i.e.  $|\phi| < 1$ . With this restriction, Proposition 1 gives us  $\bar{\sigma}^2 = \sigma^2$ ,  $T = 1$  and  $\bar{\phi} = \phi$ , while from Proposition 2 we have  $T = 1$  and  $U = 1$ . Hence, the  $MA(1)$  model is globally identified when  $|\phi| < 1$ , which is consistent with what Theorem 1 gave us as the alternative solution  $\frac{1}{\phi}$  is precluded from the space of allowable parameters when Assumption 3 is imposed.

To sum up, Theorem 1 is general in that it comprises both invertible and noninvertible  $MA(1)$  models, proving their global identification failure unless  $\phi = \pm 1$ . The less general and obtained under stronger assumptions Propositions 1 and 2 allow only for invertible  $MA(1)$  processes, in which case their global identification holds.

<sup>6</sup>Note that, since  $D$  is restricted to 1, Assumption 4 holds automatically.

Now let us consider an  $ARMA(1, 1)$  process  $y_t = \psi y_{t-1} + \phi \varepsilon_{t-1} + \varepsilon_t$ , where  $\varepsilon_t \sim N(0, \sigma^2)$ , which can be written in the state space form  $y_t = s_{t-1} + \varepsilon_t$  and  $s_t = \psi s_{t-1} + (\psi + \phi)\varepsilon_t$ . Its ABCD-representation is then  $A = \psi$ ,  $B = \psi + \phi$ ,  $M = 0$ ,  $C = 1$ ,  $D = 1$ ,  $\Sigma = \sigma^2$ . Assumption 1 holds if  $|\psi| < 1$ . As regards Assumption 2, we have  $\text{rank}(\mathcal{O}) = \text{rank}(C) = 1$  and, since  $N = (1 + \frac{\psi(\psi + \phi)}{1 - \psi^2})(\psi + \phi)\sigma^2$ , we have  $\text{rank}(\mathcal{C}) = \text{rank}(N) = 1$  iff  $\psi + \phi \neq 0$  and  $\psi\phi \neq -1$ . Since  $C = 1$ , restrictions 1) and 2) in Theorem 1 immediately give us  $T = 1$  and  $\bar{\psi} = \psi$ . Then, using 3), 4) and 5) in this theorem gives us exactly two solutions  $(Q, \bar{\psi}, \bar{\phi}, \bar{\sigma}^2) = (0, \psi, \phi, \sigma^2)$  and  $(Q, \bar{\psi}, \bar{\phi}, \bar{\sigma}^2) = (\sigma^2(\phi^2 - 1), \psi, \frac{1}{\phi}, \phi^2\sigma^2)$  for  $\phi \neq \pm 1$ , and one solution  $(Q, \bar{\psi}, \bar{\phi}, \bar{\sigma}^2) = (0, \psi, \phi, \sigma^2)$  for  $\phi = \pm 1$ . The conclusion on identification of the  $ARMA(1, 1)$  model is hence the same as of the  $MA(1)$  model – both are not globally identified at  $\phi \neq \pm 1$ . The conditions to apply Propositions 1 and 2 are also the same. This is because the Riccati equation (8) possesses the same two solutions  $\Pi = 0$  and  $\Pi = \sigma^2(\phi^2 - 1)$  so that, for Assumption 3 to hold, we need to restrict  $|\phi| < 1$ . Under this restriction,  $ARMA(1, 1)$  is globally identified.

### 3.3 Comparison with Komunjer and Ng (2011)

It is useful to juxtapose our identification analysis in these two simple examples with the approach proposed by Komunjer and Ng (2011), who also use the ABCD-representation and rely on very similar assumptions to ours. Let us start with the  $MA(1)$  case. To apply their Proposition 1-S, we need to check three assumptions. The first one (Assumption 2) is identical to our stability Assumption 1. Their Assumption 5-S corresponds to our stochastic minimality assumption 2 and in the  $MA(1)$  case leads to the same restriction  $\phi \neq 0$ . The additional restriction in Komunjer and Ng (2011) (Assumption 4-S) is left-invertibility, which holds if  $|z| > 1$  implies  $H(z)$  has full column rank, and in the context of our univariate examples is equivalent to  $H(z) = 0$  implies  $|z| \leq 1$ . In the  $MA(1)$  case we have  $H(z) = D + C(zI_n - A)^{-1}B = 1 + z^{-1}\phi = 0 \Leftrightarrow z = -\phi$ , hence left-invertibility means  $|\phi| \leq 1$ . This restriction is hence similar to that implied by our Assumption 3, which we need for the specialized Propositions 1 and 2 but not for the more general Theorem 1. As discussed above, it leads to the conclusion that  $MA(1)$  is globally identified, while in fact it is not unless noninvertible cases are ruled out.

One can proceed similarly with the  $ARMA(1, 1)$  model, for which Proposition 1-S in Komunjer and Ng (2011) requires  $|\psi| < 1$  (stability),  $\psi + \phi \neq 0$  (Assumption 5-S) and  $|\phi| \leq 1$  (left-invertibility). Recall that our analysis using Theorem 1 does not need the left-invertibility restriction, but our minimality Assumption 2 turns out to be now more restrictive than its counterpart Assumption 5-S in Komunjer and Ng (2011) as it additionally imposes  $\psi\phi \neq -1$ .<sup>7</sup> Overall, comparing all assumptions in the two identification setups for the  $ARMA(1, 1)$  model reveals that those imposed by Komunjer and Ng (2011) rule out all non-invertible cases while they are allowed in our framework except when  $\psi\phi \neq -1$ . Similarly to the  $MA(1)$  example, this has implications for the global identification analysis.

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<sup>7</sup>Note that, since  $|\psi| < 1$ , this restriction only applies to  $\phi$ 's in the non-invertibility region.

## 4 Global identification condition for structural parameters

The ABCD-representation is defined by matrices that, except for some very special cases like the simple time series models elaborated above, are very sophisticated functions of  $\theta$ . As demonstrated by Morris (2017), this is the case even for relatively simple setups like the An and Schorfheide (2007) model.<sup>8</sup> Therefore, to check identification of the vector of deep parameters, we typically cannot apply Theorem 1 directly. Hence, we proceed as in Kocięcki and Kolasa (2018) and treat those matrices in this theorem that are only implicit functions of the deep parameters as unknown, but additionally impose on them restrictions that would guarantee consistency with the underlying DSGE model structure. Following a similar approach as in Iskrev (2010), and in the spirit of the undetermined coefficient method by Uhlig (1999), these can be readily obtained by substituting the model solution (2)-(3) into model formulation (1), which we rewrite for convenience in a block form

$$\begin{bmatrix} \Gamma_0^s & \Gamma_0^p \\ \Gamma_1^s & \Gamma_1^p \end{bmatrix} \begin{bmatrix} s_t \\ p_t \end{bmatrix} = \begin{bmatrix} \Gamma_1^s & \Gamma_1^p \end{bmatrix} \mathbb{E}_t \begin{bmatrix} s_{t+1} \\ p_{t+1} \end{bmatrix} + \Gamma_2 s_{t-1} + \Gamma_3 \varepsilon_t \quad (9)$$

Using  $\mathbb{E}_t \varepsilon_{t+1} = 0$  results in the following two matrix equation restrictions

$$\Gamma_0^s A + \Gamma_0^p F - \Gamma_1^s A^2 - \Gamma_1^p F A = \Gamma_2 \quad (10)$$

$$\Gamma_1^s A B + \Gamma_1^p F B - \Gamma_0^s B + \Gamma_3 = \Gamma_0^p G \quad (11)$$

A similar operation using the original measurement equation (4) results in two other matrix restrictions, that are already available as equations (6) and (7).

We hence arrive at the following final set of conditions that have to be met by any parameter vector  $\bar{\theta}$  that is observationally equivalent to some  $\theta$

$$\bar{\Gamma}_0^s \bar{A} + \bar{\Gamma}_0^p \bar{F} - \bar{\Gamma}_1^s (\bar{A})^2 - \bar{\Gamma}_1^p \bar{F} \bar{A} = \bar{\Gamma}_2 \quad (12)$$

$$\bar{\Gamma}_1^s \bar{A} \bar{B} + \bar{\Gamma}_1^p \bar{F} \bar{B} - \bar{\Gamma}_0^s \bar{B} + \bar{\Gamma}_3 = \bar{\Gamma}_0^p \bar{G} \quad (13)$$

$$\bar{C} = \bar{H}^s \bar{A} + \bar{H}^p \bar{F} \quad (14)$$

$$\bar{D} = \bar{H}^s \bar{B} + \bar{H}^p \bar{G} + \bar{J} \quad (15)$$

$$\bar{A} = T A T^{-1} \quad (16)$$

$$\bar{C} = C T^{-1} \quad (17)$$

$$A Q A' - Q = -B \Sigma B' + T^{-1} \bar{B} \bar{\Sigma} \bar{B}' (T^{-1})' \quad (18)$$

$$A Q C' = T^{-1} \bar{B} \bar{\Sigma} \bar{D}' - B \Sigma D' \quad (19)$$

$$C Q C' = \bar{D} \bar{\Sigma} \bar{D}' - D \Sigma D' \quad (20)$$

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<sup>8</sup>See also Zadzorny (2022) on the mapping between structural and reduced-form coefficients when the solution to a DSGE model has a VAR representation in observable variables.

$$\bar{M} = M \quad (21)$$

$$Q = Q' \quad (22)$$

In this system of equations, the unknowns of interest are structural parameters  $\bar{\theta}$ , on which the following depend explicitly:  $\bar{\Gamma}_0^s, \bar{\Gamma}_0^p, \bar{\Gamma}_1^s, \bar{\Gamma}_1^p, \bar{\Gamma}_2, \bar{\Gamma}_3, \bar{\Sigma}, \bar{H}^s, \bar{H}^p, \bar{J}, \bar{M}$ . The other unknowns, namely  $\bar{A}, \bar{B}, \bar{C}, \bar{D}, \bar{F}, \bar{G}, T$  and  $Q$ , play only an auxiliary role. All remaining matrices are functions of  $\theta$ , and hence known while checking identification at this point in the parameter space. Therefore, our final global identification condition for the structural (deep) model parameters can be stated as follows

**Definition 2.** *The model given by equations (1) and (4) is globally identified if and only if all admissible solutions to system (12)-(22) are such that  $\bar{\theta} = \theta$ .*

By operating on the structural model parameters, this definition is equivalent to Definition 1, refining it such that it becomes operational.

Note that being able to solve the system of equations above analytically, i.e. giving the full set of  $\bar{\theta} \in \Theta$  that satisfy it, essentially resolves the problem of identification in a given DSGE model. However, this is not easy as equations (12)-(22) are non-linear and their number is fairly large even for small-scale models. Naturally, one can try to solve this system numerically, as it is done in a less general framework by Kocięcki and Kolasa (2018), but numerical methods can give only one solution at a time rather than their full set. Solving the identification problem hence requires analytical methods, and to this end we will use some concepts developed in computational algebraic geometry.

To apply these methods, we first need to write our model such that the coefficients on the model variables that show up in the equations of the original model formulation (1) and (4) form polynomials. In many cases this is straightforward and can be achieved by basic algebraic operations on the model equations. For example, if some coefficients in a model equation form a fraction, we can simply multiply all terms in this equation by the denominator of this fraction. Whenever this is not possible, e.g. when one parameter enters as an exponent of another, we need to define auxiliary parameters that add to the original ones, possibly replacing some of them.<sup>9</sup> We will denote the thus obtained modified parameter vector as  $\alpha$ , and will refer to its elements as semi-structural parameters, as opposed to deep parameters collected in  $\theta$ . Naturally, since the deep and semi-structural parameters are linked analytically having the solution to the global identification problem defined for  $\alpha$  makes the identification analysis for  $\theta$  easier, if not straightforward.

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<sup>9</sup>Rewriting the model equations using auxiliary parameters may be useful even if it is not necessary to obtain a polynomial structure. Take for example the New Keynesian Phillips curve  $\pi_t = \beta E_t \pi_{t+1} + \frac{(1-\xi)(1-\beta\xi)}{\xi} x_t$ , which can be easily cast in the form required by our analysis by multiplying it by  $\xi$ . Obviously, for any value of the Calvo probability  $\xi$  there exists an observationally equivalent alternative number, which lies outside the unit interval and hence should not be taken into account. However, while solving the identification problem mathematically, this alternative parametrization will be found, only to be discarded after applying economic restrictions. If we instead replace  $\xi$  in the vector of model parameters with a semi-structural parameter  $\kappa = \frac{(1-\xi)(1-\beta\xi)}{\xi}$ , this validation step can be avoided.

To see the nature of the underlying problem a bit more clearly, let us eliminate some terms in the system of equations (12)-(22) and reorganize to get

$$\bar{\Gamma}_0^s T A + \bar{\Gamma}_0^p \bar{F} - \bar{\Gamma}_1^s T A^2 - \bar{\Gamma}_1^p \bar{F} A = \bar{\Gamma}_2 T \quad (23)$$

$$\bar{\Gamma}_1^s T A \bar{B} + \bar{\Gamma}_1^p \bar{F} \bar{B} - \bar{\Gamma}_0^s T \bar{B} + \bar{\Gamma}_3 = \bar{\Gamma}_0^p \bar{G} \quad (24)$$

$$C = \bar{H}^s T A + \bar{H}^p \bar{F} \quad (25)$$

$$\bar{D} = \bar{H}^s T \bar{B} + \bar{H}^p \bar{G} + \bar{J} \quad (26)$$

$$A Q A' - Q = -B \Sigma B' + \bar{B} \bar{\Sigma} \bar{B}' \quad (27)$$

$$A Q C' = \bar{B} \bar{\Sigma} \bar{D}' - B \Sigma D' \quad (28)$$

$$C Q C' = \bar{D} \bar{\Sigma} \bar{D}' - D \Sigma D' \quad (29)$$

$$\bar{M} = M \quad (30)$$

$$Q = Q' \quad (31)$$

where  $\bar{F} = \bar{F} T$  and  $\bar{B} = T^{-1} \bar{B}$ .<sup>10</sup> We have thus turned our identification conditions into a system of polynomial equations. In this alternative formulation, the unknowns are:  $\bar{\alpha}$  (on which the following depend analytically:  $\bar{\Gamma}_0^s, \bar{\Gamma}_0^p, \bar{\Gamma}_1^s, \bar{\Gamma}_1^p, \bar{\Gamma}_2, \bar{\Gamma}_3, \bar{\Sigma}, \bar{H}^s, \bar{H}^p, \bar{J}, \bar{M}$ ), as well as matrices  $\bar{B}, \bar{D}, \bar{F}, \bar{G}, T$  and  $Q$ .

It is straightforward to derive a similar set of identification conditions for the square case, when we can use the similarity transformation defined by Proposition 2. These are

$$\bar{\Gamma}_0^s T A + \bar{\Gamma}_0^p \bar{F} T - \bar{\Gamma}_1^s T A^2 - \bar{\Gamma}_1^p \bar{F} T A = \bar{\Gamma}_2 T \quad (32)$$

$$\bar{\Gamma}_1^s T A B U + \bar{\Gamma}_1^p \bar{F} T B U - \bar{\Gamma}_0^s T B U + \bar{\Gamma}_3 = \bar{\Gamma}_0^p \bar{G} \quad (33)$$

$$C = \bar{H}^s T A + \bar{H}^p \bar{F} T \quad (34)$$

$$D U = \bar{H}^s T B U + \bar{H}^p \bar{G} + \bar{J} \quad (35)$$

$$\bar{M} = M \quad (36)$$

$$U \bar{\Sigma} U' = \Sigma \quad (37)$$

forming a system of polynomial equations in  $\bar{\alpha}, \bar{F}, \bar{G}, T$  and  $U$ .

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<sup>10</sup>Note that, since  $T$  is nonsingular, there is a one-to-one relationship between these two newly defined matrices and their parents  $\bar{F}$  and  $\bar{B}$ .

## 5 Implementation

As we have demonstrated in the previous section, the key step in solving the identification problem in a DSGE model boils down to solving a system of polynomial equations. In our implementation we draw on the concept of a Gröbner basis. Intuitively, calculating it is analogous to Gaussian elimination in systems of linear equations, and it is entirely analytical. There exist many algorithms that produce a Gröbner basis in finitely many steps and, since the first algorithm proposed by Buchberger in the 1960s, enormous progress in computational efficiency has been made. Below we first offer a brief introduction to the key concepts. Readers interested just in application of our framework may want to skip this part and move directly to Section 5.2. Those interested in more theoretical detail, we refer to widely suggested introductory textbooks on computational algebraic geometry (and Gröbner basis in particular) by Cox et al. (1997) and Cox et al. (2005). An excellent introduction in the context of finding all equilibria in economic models can be found in Kubler et al. (2014).

### 5.1 Gröbner basis

#### 5.1.1 Theoretical foundations

Let  $\mathbb{K}$  denote any field. For us, the most important fields are  $\mathbb{Q}$ , i.e. that of rational numbers, the field of real numbers, i.e.  $\mathbb{R}$  and the field of complex numbers, i.e.  $\mathbb{C}$ . In addition, let us denote as  $\bar{\mathbb{K}}$  an algebraically closed field containing  $\mathbb{K}$ . Without going into details, one can think of  $\bar{\mathbb{K}}$  as  $\mathbb{C}$ . The set of polynomials in variables  $x_1, \dots, x_l$  with coefficients in  $\mathbb{K}$  will be denoted  $\mathbb{K}(x_1, \dots, x_l)$ . Each polynomial equation is a finite sum of terms  $cx_1^{d_1}x_2^{d_2} \dots x_l^{d_l}$ , where  $c$  is a coefficient (in  $\mathbb{K}$ ) and  $x_1^{d_1}x_2^{d_2} \dots x_l^{d_l}$  is called a monomial, where each  $d_i$  is a non-negative integer. The degree of a monomial is  $d_1 + \dots + d_l$ , and the degree of a polynomial equation is the maximum of the degrees of its all monomials.

Suppose we have a set of  $s$  polynomials  $f_1, f_2, \dots, f_s \in \mathbb{K}(x_1, \dots, x_l)$ . Then, a variety  $V$  is defined to be a set of all solutions to  $f_1 = 0, f_2 = 0, \dots, f_s = 0$ , i.e.  $V(f_1, \dots, f_s) = \{(a_1, \dots, a_l) \in \bar{\mathbb{K}}^l | f_1 = 0, \dots, f_s = 0\}$ . Of course, the initial polynomials  $f_1, \dots, f_s$  only represent the variety. There are many other alternative sets of polynomials, some of which could do a better job in the sense of the ease with which the underlying solutions can be read. In particular, this opens a way to a Gröbner basis. To this end, define an ideal generated by  $f_1, \dots, f_s$  as  $I = \langle f_1, \dots, f_s \rangle = \{u_1f_1 + \dots + u_sf_s | u_i \in \mathbb{K}(x_1, \dots, x_l), i = 1, \dots, s\}$ . The ideal is just a set of all polynomials that may be written as a weighted sum of all initial polynomials (called its generators), in which the coefficients (weights) are polynomials themselves. For example  $x^2 \in \langle x - y^2, xy \rangle$ , since  $x^2 = x \cdot (x - y^2) + y \cdot (xy)$ .<sup>11</sup> What makes the ideal useful is that  $V(f_1, \dots, f_s) = V(I)$ , i.e. the solution set of the initial finite system of polynomials and that of an ideal generated by these polynomials (i.e. an infinite system) are the same. To realize it, note that a solution of  $I$  is also a solution of  $f_1 = 0, f_2 = 0, \dots, f_s = 0$  since

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<sup>11</sup>In what follows, when we will present examples with small number of variables we will sometimes use  $x, y, z$ , etc., instead of  $x_1, x_2, x_3$ , etc.

$f_i \in I$ , for each  $i = 1, \dots, s$ . (just put one  $u_i = 1$  and the remaining  $u_j = 0$ , for all  $j \neq i$ ). On the other hand, if  $a^* = (a_1^*, \dots, a_l^*)$  is a solution to  $f_1 = 0, f_2 = 0, \dots, f_s = 0$ , then when evaluated at this  $a^*$ ,  $u_1 f_1 + \dots + u_s f_s = 0$  for all  $u_i \in \mathbb{K}(x_1, \dots, x_l)$ . Evidently, any ideal can have different generators. As a matter of fact, if  $\langle f_1, \dots, f_s \rangle = \langle f'_1, \dots, f'_{s'} \rangle$ , then  $V(f_1, \dots, f_s) = V(I) = V(f'_1, \dots, f'_{s'})$ . Hence, the solutions to  $f_1 = 0, \dots, f_s = 0$  and to  $f'_1 = 0, \dots, f'_{s'} = 0$  are the same. In the essence, what defines the solution set is the ideal and not the initial polynomials. The main idea of a Gröbner basis is to find alternative generators that represent the ideal in a “better” way. For example, in the case of linear polynomials (equations), this “better” way is to find their row echelon form. Importantly, by the Hilbert basis theorem, each ideal must be generated by finite number of polynomials, hence the algorithmic methods to find a “better” representation of the variety may be safely applied.

Before we can define (and obtain) a Gröbner basis, we have to take a stand on the ordering of monomials since every algorithm to compute the basis must involve polynomial divisions. It is not sufficient to set up the ordering of the variables  $x_i$  in  $\mathbb{K}(x_1, \dots, x_l)$ , indicating implicitly  $x_1 > x_2 > \dots > x_l$ , since this would still leave room to write a polynomial with different orders of summation. An ordering is a rule that allows for a unique placement of terms in a polynomial. It turns out that the chosen ordering greatly influences the ultimate Gröbner basis, and some orderings are particularly useful. For our purposes, the most important ordering is the so-called lexicographic (in short, lex) ordering, i.e.  $x_1^{d_1} x_2^{d_2} \dots x_l^{d_l} >_{lex} x_1^{e_1} x_2^{e_2} \dots x_l^{e_l}$  in  $\mathbb{K}(x_1, \dots, x_l)$ , with variables implicitly ordered as  $x_1 > x_2 > \dots > x_l$ , if  $d_1 = e_1, \dots, d_m = e_m$  and  $d_{m+1} > e_{m+1}$  (where possibly  $m = 0$ ). For example,  $x_1 x_2^2 x_3 >_{lex} x_1 x_2^2 x_4^3$  since  $d_3 = 1 > e_3 = 0$ . Let us define  $x^d := x_1^{d_1} x_2^{d_2} \dots x_l^{d_l}$ . If we choose a monomial ordering, each polynomial may be written uniquely as  $f = cx^d + \dots$ . Then  $x^d$  is called the leading monomial and  $cx^d$  is the leading term, which we denote  $LT(f) = cx^d$ . For example, consider  $f = -\frac{2}{3}x_1^3 x_2 - \frac{7}{2}x_1^2 x_2^2 + 2x_1 x_2^3 + \frac{1}{2}x_3^4 \in \mathbb{Q}(x_1, x_2, x_3)$ . This polynomial is written in lex ordering, with  $LT(f) = -\frac{2}{3}x_1^3 x_2$ , and the leading monomial  $x_1^3 x_2$ . Let  $LT(I)$  be the set of all leading terms of elements of  $I$  i.e.  $LT(I) = \{cx^d \mid \text{there exists } f \in I \text{ with } LT(f) = cx^d\}$ . Let us define the ideal generated by the elements of  $LT(I)$  as  $\langle LT(I) \rangle$ . Then we have a definition.

**Definition 3.** Assuming any monomial ordering,  $g_1, \dots, g_t \in I$  is a Gröbner basis for ideal  $I$  if  $\langle LT(g_1), \dots, LT(g_t) \rangle = \langle LT(I) \rangle$

Equivalently, we can say that polynomials  $g_1, \dots, g_t \in I$  constitute the Gröbner basis for ideal  $I$  if the leading term of any (nonzero) polynomial in  $I$  is divisible by the leading term of one of  $g_1, \dots, g_t$ .<sup>12</sup> The practical meaning of this is that if  $g_1, \dots, g_t \in I$  is the Gröbner basis for ideal  $I$ , then any  $f \in I$  may be written as  $f = a_1 g_1 + a_2 g_2 + \dots + a_t g_t$ , where each  $a_i$  is a polynomial in  $\mathbb{K}(x_1, \dots, x_l)$ . Hence  $V(f_1, \dots, f_s) = V(I) = V(g_1, \dots, g_t)$ . In other words,  $g_1, \dots, g_t$  are generators for  $I$ , every (nonzero)  $I$  possesses a Gröbner basis, and solutions to  $g_1 = 0, \dots, g_t = 0$  and to the initial polynomials  $f_1 = 0, \dots, f_s = 0$  are the same. When there is only a finite number of solutions, the underlying ideal is called zero-dimensional.

The Gröbner basis contains a lot of information about the solutions set of the initial polynomial system. For example, the initial system does not have any solution if and only if the Gröbner basis

<sup>12</sup>A nonzero term  $c_d x^d$  is divisible by a nonzero term  $c_e x^e$  if  $d_i \geq e_i$  for all  $i$ .



contains only 1. Further, whether an ideal is zero-dimensional or not is explicitly “coded” in the Gröbner basis and can be easily read off. The initial system of polynomials possesses a finite number of solutions (i.e.  $I$  is zero-dimensional) if and only if for every variable  $x_i$  there exists a polynomial in the Gröbner basis such that its leading monomial is equal to  $x_i^m$ , for some  $m > 0$ . Importantly, calculation of a Gröbner basis is analytical, i.e. numerical approximations are not involved.

The fact that in our exposition we confined to lex orderings is not a coincidence for then we have the next result of great practical importance.

**Proposition 3.** *Let  $I$  be a zero-dimensional ideal and  $g_1, \dots, g_t \in I$  be the Gröbner basis for  $I$  with respect to the lex ordering with  $x_1 < x_2 < \dots < x_l$ . Then it must be that  $t \geq l$  and we can order  $g_1, \dots, g_t$  such that  $g_1$  contains only  $x_1$ ,  $g_2$  can contain only variables  $x_1$  and  $x_2$  and the leading monomial of  $g_2$  is  $x_2^{m_2}$ , for some  $m_2 > 0$ ,  $g_3$  can contain only variables  $x_1, x_2$  and  $x_3$  and the leading monomial of  $g_3$  is  $x_3^{m_3}$ , for some  $m_3 > 0$ , and so on until  $g_l$ , which can contain all variables and its leading monomial is  $x_l^{m_l}$ , for some  $m_l > 0$ .*

Hence, using the lex ordering for a zero-dimensional ideal, the resulting Gröbner basis becomes a “triangularized” system of equations.<sup>13</sup>

### 5.1.2 Examples

To illustrate the usefulness of Proposition 3, we offer several simple examples. Consider first the case  $t = l$

$$\begin{aligned} x^5 + y^2 + z^2 - 4 &= 0 \\ x^2 + 2y^2 - 5 &= 0 \\ xz - 1 &= 0 \end{aligned} \tag{38}$$

The Gröbner basis for (38) with respect to lex ordering with  $z < y < x$  is

$$\begin{aligned} 2z^7 - 3z^5 - z^3 + 2 &= 0 \\ 4y^2 - 2z^5 + 3z^3 + z - 10 &= 0 \\ 2x + 2z^6 - 3z^4 - z^2 &= 0 \end{aligned} \tag{39}$$

Since the leading monomials in the Gröbner basis comprise  $z^7$ ,  $y^2$  and  $x$ , the ideal generated by (38) is zero-dimensional. Bearing in mind that solutions of (38) and (39) are the same, solving (38) amounts to solving the first equation in (39) for  $z$ . For every such obtained solution with respect to  $z$ , we get two solutions for  $y$  using second equation in (39), and one solution for  $x$  using the last equation in (39).

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<sup>13</sup>Under some further conditions, the Gröbner basis for a zero-dimensional ideal in lex ordering is  $\{q_1(x_1), x_2 - q_2(x_1), x_3 - q_3(x_1), \dots, x_l - q_l(x_1)\}$ , where every  $q_i(x_1)$  is a univariate polynomial in  $x_1$ . This is the content of the Shape Lemma, see e.g. Becker et al. (1994), which was exploited in Kubler et al. (2014). This is a particularly convenient setup since, by solving  $q_1(x_1)$  for  $x_1$  and plugging this solution in the remaining equations, we immediately obtain the total set of solutions. As useful as it may appear, this concept plays no role in our approach as the Gröbner bases in our identification analysis typically do not have to possess the Shape Lemma structure.

On the other hand, if  $t > l$ , in order to find the solutions to the whole system of equations we have to check whether the solutions of  $g_1 = 0, \dots, g_l = 0$  still fulfill  $g_{l+1} = 0, \dots, g_t = 0$ , and discard those solutions which do not satisfy the latter. To this end, consider the following system

$$\begin{aligned}x^2 + xy + 2x + y - 1 &= 0 \\x^2 + 3x - y^2 + 2y - 1 &= 0\end{aligned}\tag{40}$$

The Gröbner basis in lex ordering with  $y < x$  is

$$\begin{aligned}y^2 - 1 &= 0 \\x^2 + 3x + 2y - 2 &= 0 \\xy - x - y + 1 &= 0\end{aligned}\tag{41}$$

Since the leading monomials include  $y^2$  and  $x^2$ , the underlying ideal is zero-dimensional. From the first equation we conclude  $y = \pm 1$ . If  $y = 1$ , the second equation becomes  $x(x+3) = 0$ , while the last one is  $x = x$  (i.e. not binding). Hence the first two solutions are  $(x, y) = (0, 1)$  and  $(x, y) = (-3, 1)$ . If we put  $y = -1$ , then the second equation gives two solutions for  $x$  i.e.  $x = 1$  or  $x = -4$ . However, only  $x = 1$  is consistent with the last equation. Hence, the third solution is  $(x, y) = (1, -1)$ . As we discuss later, in practice we do not have to find the full set of solutions of the Gröbner basis for a zero-dimensional ideal manually since every computer algebra package has a dedicated function to do this.

Let us now turn to the case of ideals that are not zero-dimensional, which means that the analyzed system has infinite number of solutions, and which will typically represent local identification failure in a model identification analysis. While in this case Proposition 3 cannot be applied, the Gröbner basis in lex ordering might also be useful to shed some light on the solution nature. To illustrate this point, consider the following system

$$\begin{aligned}x^2 - y^2 + x + y - z &= 0 \\x^2 + 2y^2 - 2x + y - z &= 0 \\x^3 - x^2z - xy^2 - 2y^2z + x^2 + xy + xz - yz + z^2 &= 0\end{aligned}\tag{42}$$

Computing the Gröbner basis in lex ordering with  $z < y < x$  for (42) results in two equations

$$\begin{aligned}y^4 + y - z &= 0 \\x - y^2 &= 0\end{aligned}\tag{43}$$

We immediately see that the ideal generated by (42) is not zero-dimensional since there is no  $z^m$ , for some  $m > 0$ , among the leading monomials. However, the Gröbner basis is still useful as it points that setting  $y$  to any number will result in unique solutions with respect to  $x$  and  $z$ , but fixing either  $z$  or  $x$  in general does not imply uniqueness for the remaining variables. Hence, if  $x, y, z$  were the parameters of a model, then fixing  $y$  would identify it. It is not difficult to anticipate that analogous reasoning will be key to achieve identification in non-identified DSGE models, which we will discuss in Section 6.

## 5.2 Computation

For all our calculations that follow, we use SINGULAR, a free and open source computer algebra system specialized in polynomial calculations, see at [www.singular.uni-kl.de](http://www.singular.uni-kl.de). It has implemented many routines for calculation of the Gröbner basis, which is useful as there is no single algorithm that beats in terms of computational efficiency all alternatives for all possible cases. SINGULAR can be considered a repository of most state-of-the-art algorithms, with active community of users sharing their experience in approaching various problems.

Importantly, SINGULAR has built-in routines that compute all solutions to zero-dimensional ideals once the Gröbner basis is obtained, also in the so-called overdetermined cases when the number of equations exceeds the number of variables. To deal with them effectively, it exploits the approaches presented in Lazard (1992) or Möller (1993). The idea is to decompose the original zero-dimensional ideal so that the solution set of the initial polynomials will be the disjoint (finite) union of solutions to some smaller systems of  $l$  equations in  $l$  variables. This leads to the so-called triangular decomposition. See Kübler et al. (2014), section 2.2.3, for some intuition. The computation of all solutions is numerical but can be done with arbitrary precision.

In order to appreciate the great potential of the Gröbner basis in tracing identification problems, let us consider a simple New Keynesian model used in a similar context e.g. in Koop et al. (2013). The model is easy to handle as, in contrast to typical DSGE models that we discuss in Section 6, it does not include state variables. More specifically, the model is given by the following three equations

$$R_t = \psi\pi_t + \varepsilon_{1,t} \quad (44)$$

$$x_t = \mathbb{E}_t x_{t+1} - \tau(R_t - \mathbb{E}_t \pi_{t+1}) + \varepsilon_{2,t} \quad (45)$$

$$\pi_t = \beta\mathbb{E}_t \pi_{t+1} + \kappa x_t + \varepsilon_{3,t} \quad (46)$$

which describe the evolution of three endogenous variables: output  $x_t$ , inflation  $\pi_t$  and the interest rate  $R_t$ . The model dynamics is driven by three i.i.d. shocks collected in vector  $\varepsilon_t = [\varepsilon_{1,t}, \varepsilon_{2,t}, \varepsilon_{3,t}]'$ , where  $\varepsilon_t \sim i.i.d. N(0, \Sigma)$  and  $\Sigma = \text{diag}(v_1, v_2, v_3)$ , with  $v_i$  for  $i = 1, 2, 3$  denote the variances of the shocks. The model can be cast in our basic form (1) with:

$$\Gamma_0 = \begin{bmatrix} 1 & 0 & -\psi \\ \tau & 1 & 0 \\ 0 & -\kappa & 1 \end{bmatrix}; \quad \Gamma_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & \tau \\ 0 & 0 & \beta \end{bmatrix}; \quad \Gamma_2 = 0; \quad \Gamma_3 = \mathbf{I}_3$$

We assume that all endogenous variables are observable, demeaned and that there are no measurement errors, hence  $y_t = [R_t \ x_t \ \pi_t]'$ ,  $M = 0$ ,  $H = \mathbf{I}_3$  and  $J = 0$ . The 7-dimensional vector of deep parameters is  $\theta = [\psi \ \tau \ \beta \ \kappa \ v_1 \ v_2 \ v_3]'$ .

The model admits one stable solution, which can be written as  $\Gamma_0 y_t = \varepsilon_t$ . Since then  $y_t \sim i.i.d. N(0, \Gamma_0^{-1} \Sigma (\Gamma_0^{-1})')$ , there is no need to describe the dynamics in terms of the state-space

system. The model is globally identified at  $\Gamma_0, \Sigma$  if  $\bar{\Gamma}_0^{-1}\bar{\Sigma}(\bar{\Gamma}_0^{-1})' = \Gamma_0^{-1}\Sigma(\Gamma_0^{-1})'$  implies  $\bar{\Gamma}_0 = \Gamma_0$  and  $\bar{\Sigma} = \Sigma$ . The fact that the discount factor parameter  $\beta$  is not identified is obvious as it does not show up in the solution. Hence, in what follows, we set  $\beta = 0.995$ , which is still necessary for stability. Less obvious but also well known result is that all the remaining parameters are globally unidentified too, see e.g. Rubio-Ramírez et al. (2010). To demonstrate it using the Gröbner basis, let us first write the identification condition in an equivalent form  $\bar{\Gamma}'_0\bar{\Sigma}^{-1}\bar{\Gamma}_0 = \Gamma'_0\Sigma^{-1}\Gamma_0$  and define auxiliary parameters  $(\alpha_1, \alpha_2, \alpha_3) = (v_1^{-1}, v_2^{-1}, v_3^{-1})$  that replace the three variances in  $\theta$ , so that the vector of semi-structural parameters is  $\alpha = [\psi \ \tau \ \beta \ \kappa \ \alpha_1 \ \alpha_2 \ \alpha_3]'$ . Obviously, the mapping between  $\alpha$  and  $\theta$  is unique and straightforward.

Let us consider the following benchmark parameter values at which we check identification:  $\psi = 2$ ,  $\tau = 0.5$ ,  $\kappa = 0.6$ ,  $v_1 = 0.04$ ,  $v_2 = 0.1$ ,  $v_3 = 0.15$ .<sup>14</sup> The Gröbner basis, calculated with SINGULAR function `std` assuming lex ordering consistent with that we used to define  $\alpha$ , is then

$$\begin{aligned}
237\bar{\alpha}_3^2 - 2360\bar{\alpha}_3 + 5200 &= 0 \\
325\bar{\alpha}_2 - 237\bar{\alpha}_3 - 1670 &= 0 \\
980\bar{\alpha}_1 - 237\bar{\alpha}_3 - 22920 &= 0 \\
1300\bar{\kappa} + 237\bar{\alpha}_3 - 2360 &= 0 \\
4900\bar{\tau} + 237\bar{\alpha}_3 - 4030 &= 0 \\
150\bar{\psi} + 3\bar{\alpha}_3 - 320 &= 0
\end{aligned} \tag{47}$$

As we emphasized before, the Gröbner basis conveys a lot of useful information. First of all, inspecting it reveals that our identification condition possesses exactly two solutions. This is because the first (quadratic) equation in (47) has two solutions, for each of which we can recursively and uniquely obtain all other parameters using the remaining equations. We do not need to do this manually as SINGULAR contains the useful procedure `solve` to find them explicitly. Obviously, the first solution is the point at which we check identification while the second one (reported here up to four decimal digits) is:  $\bar{\tau} = 0.6633$ ,  $\bar{\kappa} = 1.2154$ ,  $\bar{\psi} = 2.0675$ ,  $\bar{\alpha}_1 = \bar{v}_1^{-1} = 24.1837$ ,  $\bar{\alpha}_2 = \bar{v}_2^{-1} = 7.5385$ ,  $\bar{\alpha}_3 = \bar{v}_3^{-1} = 3.2911$ , which still results in the unique stable solution for our choice of  $\beta$ . Note that numerical approximation “starts here” only because the solution to the quadratic equation involves approximation. However, the solution precision can be set in SINGULAR at an arbitrary level.

Importantly, the first equation suggests that if we fix  $\bar{\alpha}_3 = \alpha_3$  (i.e.  $\bar{v}_3 = v_3$ ), the remaining two variances, as well as  $\bar{\kappa}$ ,  $\bar{\tau}$  and  $\bar{\psi}$  are unique, hence the model is globally identified. As a matter of fact, we can fix any of the shock variances to achieve the same goal. This can be confirmed by computing the Gröbner basis with lex ordering such that the variance that we want to fix comes last. Obviously, the same conclusion can be achieved by using Gröbner basis (47), combined with the desired restriction, and by running command `solve`.

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<sup>14</sup>In principle, SINGULAR can calculate the Gröbner basis for symbolic  $\alpha$  rather than its particular value. This can be useful in simple examples like the one considered here, but not practical in typical DSGE models, in which the solution matrices do not have analytical representation.

### 5.3 Checking global identification

By calculating the Gröbner basis for a given DSGE model, we obtain a formally proved verdict about global identification of its semi-structural parameters. To complete the identification analysis about the deep model parameters, we need to use the restrictions mapping  $\bar{\theta}$  into  $\bar{\alpha}$ . In particular, this step also allows us to rule out those observationally equivalent semi-structural parameter sets that violate the restrictions imposed on them by the deep parameters.

These restrictions can be of two types. One concerns possible remaining dependencies between the semi-structural parameters, which are imposed by the deep parameters and which were ignored while defining the former. The second type of restrictions is related to the range of admissible values of  $\bar{\theta}$  summarized by their space  $\Theta$ , which may also impose restrictions on  $\bar{\alpha}$ . Accommodating such restrictions directly in the existing algorithms used to calculate the Gröbner basis is not easy, and hence they have to be verified ex post. Admittedly, this part of our identification analysis is model dependent and cannot be easily automated. As we demonstrate in our examples, since the links between  $\alpha$  and  $\theta$  are analytical, this step can be straightforward, though nothing guarantees that it will be such for all possible DSGE models. However, even if the mapping between  $\alpha$  and  $\theta$  is too sophisticated to handle analytically, applying our framework can still greatly simplify the underlying identification problem, making it easier to analyze with numerical methods.

To summarize, a complete identification analysis in our framework can proceed as follows.

1. Write the model such that Assumptions 1 and 2 are satisfied for selected parameter vector  $\theta$  and the coefficients in the model equations (1) and (4) form polynomials. If necessary or convenient, define auxiliary parameter vector  $\alpha$ .
2. Solve the model at  $\theta$  to obtain matrices  $A$ ,  $B$ ,  $F$  and  $G$ .
3. Calculate the Gröbner basis associated with identification conditions (23)-(31), or (32)-(37) if Assumptions 3 and 4 hold.
4. If the obtained Gröbner basis suggests multiple solutions, use the mapping between  $\bar{\alpha}$  and  $\bar{\theta}$  to rule out those resulting in  $\bar{\theta} \notin \Theta$  (and in particular those for which  $\bar{\theta} \in \emptyset$ ).
5. If at least one of the alternative solutions  $\bar{\theta} \neq \theta$  remains, the model is not globally identified at  $\theta$ .
6. If instead the Gröbner basis implies only one admissible solution, then the model is globally identified at  $\theta$  if and only if the mapping between  $\bar{\alpha}$  and  $\bar{\theta}$  is unique (for  $\bar{\theta} \in \Theta$ ).

This analysis can be further refined to distinguish between global and local identification failure. Again, doing it at the level of mapping between  $\alpha$  and  $\theta$  boils down to checking if a possible non-uniqueness of the mapping imply a finite number or infinitely many admissible  $\theta$ 's consistent with a given  $\alpha$ . In the former case, the model is globally unidentified, but local identification holds. Similar information at the level of semi-structural parameters is coded in the Gröbner basis. If it implies multiple admissible solutions, but is zero-dimensional, the model is locally identified.

The two key computational steps 2 and 3 can be automated by preprocessors similar to Dynare (Adjemian et al., 2011). As discussed above, calculating the Gröbner basis (step 3) requires specialized software like SINGULAR. As our procedure requires highly accurate inputs, we recommend solving the model (step 2) with procedure `nt_solve` in SINGULAR, which solves non-linear systems of equations with arbitrary precision, and which can take as starting values the model solution obtained with standard software to solve DSGE models (like Dynare).<sup>15</sup> The remaining steps are model dependent and may require manual input. As regards step 1, one may need to remove redundant state variables if the original model formulation does not meet Assumption 2, which may be tough to automate.<sup>16</sup> Rewriting the model equations such that the coefficients in the model equations form polynomials may also be best done manually. It might be also not straightforward to write a general code that examines the mapping between the deep and semi-structural parameters if the latter need to be defined (step 4).

At this point, it is useful to summarize what makes our identification framework distinct from those used in other fields. All works that apply Gröbner basis in the context of state-space models cited in the Introduction (for survey and comparison of various approaches, see also ch. 2 in Walter and Pronzato, 1997; Chis et al., 2011; Bates et al., 2019) use a continuous time setup (ordinary differential equations) and deal with the so-called structural identification. Structural identification concerns the idealized framework in which observables are measured without noise and input is known.<sup>17</sup> It answers the question whether parameters can be uniquely recovered assuming some given noise-free input-output data. In this sense, structural identification is only necessary for our analysis. Another distinct feature of our approach is that we successfully circumvent the problem that analytical mapping between the deep and ABCD-representation parameters is unknown. Beyond the state-space framework, our important contribution is the practical insight that the identification problem in DSGE models essentially boils down to solving a system of polynomial equations.

Before we proceed, let us discuss one limitation of our approach related to the current hardware restrictions. As we have already mentioned, it has been formally proved that computing a Gröbner basis requires a finite number of iterations. However, these computations can be very memory intensive. Even when the resulting Gröbner basis is small, the polynomials generated at intermediate steps can be very large so that the computing unit runs out of memory. Unfortunately, there is no easy way to judge how complicated the calculations will be just by looking at the original system. It certainly helps if the polynomials consist of a small number of monomials and their degree is also small. Inspecting our identification conditions (23)-(31) reveals that they generate monomials of relatively small degree, not exceeding three but for those that arise from possibly non-linear dependence of matrices  $\Gamma_0$ - $\Gamma_3$  on  $\theta$  (or  $\alpha$ ). The polynomials are even simpler in the square case, when conditions (32)-(37) can be used. Moreover, matrices  $\Gamma_0$ - $\Gamma_3$  are typically sparse, especially in

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<sup>15</sup>The use of arbitrary precision is also strongly favored in the identification analysis by Qu and Tkachenko (2022).

<sup>16</sup>See Komunjer and Ng (2011) for how this can be necessary and how it can be done in the case of more sophisticated models like Smets and Wouters (2007).

<sup>17</sup>Roughly speaking, this would amount to assume in our setup that all shocks are observable and there are no measurement errors.

larger DSGE models, which we will see in the examples below. These features are probably why our identification framework can work even for medium-sized models. It is also clear that the size of the model matters, and especially the number of state variables and shocks as they determine the dimensions of matrices  $\bar{B}$ ,  $\bar{D}$ ,  $\bar{F}$ ,  $\bar{G}$ ,  $T$  and  $Q$  that need to be solved for. For example, as we will show in the next section, we can easily solve the identification problem in a variant of the Smets and Wouters (2007) model in which the output gap is defined as deviation of output from trend. If we instead consider the original formulation, which includes a flexible price block with additional six state variables, our framework cannot handle it anymore due to memory constraints.

## 6 Examples

We demonstrate the working of our identification framework with several examples of DSGE models, for which, and unlike in the Koop et al. (2013) model considered in Section 5.2, we need to use the ABCD-representation. The first one is based on Cochrane (2011), and its simplicity allows us to show in detail the key steps of our analysis, including the use of the Gröbner basis. We next exploit a small-scale model by An and Schorfheide (2007), AS henceforth, modified to allow for correlation between government spending and productivity as in Herbst and Schorfheide (2016). This is a very instructive example as it allows to nest various non-trivial types of identification issues, including the case when the model is only locally (but not globally) identified. Solving the identification problem in these models using our approach takes only a fraction of a second.<sup>18</sup> We then move to two popular medium-sized models represented by Smets and Wouters (2007) and Justiniano and Preston (2010), where computational time can be non-negligible and the memory constraints mentioned in the previous section can become binding.

### 6.1 Cochrane model

#### 6.1.1 Model summary and its analytical solution

Consider a very simple model

$$i_t = \mathbb{E}_t \pi_{t+1} \tag{48}$$

$$i_t = \phi \pi_t + x_t \tag{49}$$

where  $i_t$  is the nominal interest rate,  $\pi_t$  denotes inflation and  $x_t$  is a monetary policy shock that follows a stationary AR(1) process. The first equation can be interpreted as the log-linearized Fisher relationship, while the second as a simple monetary policy feedback rule. We restrict here our attention to the case of determinacy so that  $\phi > 1$ .

Substituting out  $i_t$  and writing the process driving  $x_t$  explicitly leads to the following system

$$x_t = \rho x_{t-1} + \varepsilon_t \tag{50}$$

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<sup>18</sup>All computation is done using a unit with CPU speed 2.90 GHz and 16 GB of RAM memory.

$$\phi\pi_t + x_t = \mathbb{E}_t\pi_{t+1} \quad (51)$$

where  $|\rho| < 1$  and  $\varepsilon_t \sim N(0, v)$ , with  $v > 0$  denoting the variance. This system can be easily cast into form (1), with  $s_t = x_t$ ,  $p_t = \pi_t$ ,  $\theta = [\rho \ \phi \ v]'$  and

$$\Gamma_0 = \begin{bmatrix} 1 & 0 \\ 1 & \phi \end{bmatrix}; \quad \Gamma_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}; \quad \Gamma_2 = \begin{bmatrix} \rho \\ 0 \end{bmatrix}; \quad \Gamma_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad \Sigma = v$$

Note that all deep parameters collected in  $\theta$  enter the model equations linearly so that we do not need to rewrite them using semi-structural parameters, which we can formally write as  $\alpha = \theta$ . If the only observable variable is inflation, i.e.  $y_t = \pi_t$ , and there is no measurement error nor intercept in the observation equation, we have

$$M = 0; \quad H = \begin{bmatrix} 0 & 1 \end{bmatrix}; \quad J = 0$$

The model is simple enough to have an analytical solution, which, given the restriction on  $\phi$ , is uniquely given by formulas (2)-(3) with the following coefficients

$$A = \rho; \quad B = 1; \quad F = -\frac{\rho}{\phi - \rho}; \quad G = -\frac{1}{\phi - \rho}$$

Note that under the stated conditions  $\phi - \rho > 0$ . We also obviously have  $C = F$  and  $D = G$ . This solution implies that the observable variable can be written as an AR(1) process

$$y_t = \rho y_{t-1} - \frac{1}{\phi - \rho} \varepsilon_t \quad (52)$$

Having such an analytical solution, the identification analysis is straightforward and we can immediately conclude that, of the three model parameters, only  $\rho$  is globally identified while  $\phi$  and  $v$  cannot be separately identified.

### 6.1.2 Calculating the Gröbner basis

To demonstrate the working of our framework, suppose now that, as it is typically the case, we do not know the analytical solution of the model, so that  $A$ ,  $B$ ,  $C$  and  $D$  are just numbers. Let us apply our framework at a generic point  $\theta = [0.8 \ 1.8 \ 1]'$  so that  $A = 0.8$ ,  $B = 1$ ,  $C = -0.8$  and



$D = -1$ . The identification conditions (23)-(31) can then be written as follows

$$\begin{aligned}
0.8T &= \bar{\rho}T \\
0.8T + \bar{\phi}\bar{F} - 0.8\bar{F} &= 0 \\
-T\bar{B} + 1 &= 0 \\
\bar{F}\bar{B} - T\bar{B} &= \bar{\phi}\bar{G} \\
-0.8 &= \bar{F} \\
\bar{D} &= \bar{G} \\
0.64Q - Q &= -1 + (\bar{B})^2\bar{v} \\
-0.64Q &= \bar{B}\bar{v}\bar{D} + 1 \\
0.64Q &= (\bar{D})^2\bar{v} - 1
\end{aligned} \tag{53}$$

where the last one was omitted as it becomes an identity when the dimension of  $Q$  is one. The unknown variables are:  $\bar{\rho}$ ,  $\bar{\phi}$ ,  $\bar{v}$ , as well as  $\bar{B}$ ,  $\bar{D}$ ,  $\bar{F}$ ,  $\bar{G}$ ,  $T$  and  $Q$ , all of which are one-dimensional objects. The identification conditions are hence given by a system of polynomial equations of degree three. Finding its all solutions is not straightforward even in this simple case. We will show now how this goal can be achieved by calculating the Gröbner basis of the ideal generated by these polynomials.

As we mentioned in the previous section, defining a Gröbner basis involves ordering of monomials, which allows for a unique placement of terms in each polynomial. In applications like ours, the most convenient one is the so-called lexicographic (lex) ordering, applied to variables arranged such that the objects of interest, which are the model parameters  $\bar{\rho}$ ,  $\bar{\phi}$  and  $\bar{v}$ , come first. The sequence in which we listed the unknown variables in the previous paragraph meets this criterion, so we use it here. After applying the lex ordering to our polynomials, we obtain the following Gröbner basis

$$\begin{aligned}
0.64Q^2 + Q &= 0 \\
TQ + 0.8Q &= 0 \\
\bar{G}Q &= 0 \\
\bar{G}T + 0.64Q + 1 &= 0 \\
\bar{F} + 0.8 &= 0 \\
\bar{D} - \bar{G} &= 0 \\
\bar{B} + \bar{G} - 0.8Q &= 0 \\
\bar{v} - T^2 + 0.2304Q &= 0 \\
\bar{\phi} - T - 0.8 &= 0 \\
\bar{\rho} - 0.8 &= 0
\end{aligned} \tag{54}$$

where all numbers showing up in the equations above are exact rational numbers (even though we

present them using a decimal notation) as they are derived by analytical algebraic operations that do not involve any numerical approximation.

### 6.1.3 Identification analysis

One important thing to note is a triangular structure of the obtained Gröbner basis. The first of polynomials includes only  $Q$ , the second adds to it  $T$ , the third may additionally contain  $\bar{G}$ , and so on until the model parameters are finally added. This is exactly what makes finding all solutions of the system of polynomial equations (54) easy, in contrast to the original set of identification conditions (53), and we know that the solutions are exactly the same. Naturally, the particular sequence in which the unknown variables add to this triangle is no coincidence, but simply reflects the ordering that we have chosen. As we will show now, this often allows for straightforward conclusions on identification of the model parameters even without having to solve for all other objects.

In our particular example, we immediately see that all possible solutions must be such that  $\bar{\rho} = 0.8 = \rho$ , hence this parameter is globally identified at the  $\theta$  we consider. As regards the other two parameters, they depend on  $Q$  and  $T$ , which are fully determined by the first two equations in (54). From the first one we obtain that  $Q = -1.5625$  or  $Q = 0$ . The first case leads to  $T = -0.8$  and further to  $\bar{\phi} = 0$ , which violates the restriction imposed on this parameter, and hence can be ruled out. If instead  $Q = 0$ , the second equation does not put any restriction on  $T$ , and hence  $\bar{v}$  and  $\bar{\phi}$  are not identified. For  $T = 1$  we obtain  $\bar{\theta} = \theta$ , but any deviation of  $T$  from unity results in an alternative  $\bar{\theta}$  that is observationally equivalent to  $\theta$ . This deviation can be arbitrarily small, which means that the identification failure is local.

A useful feature of our approach is that having the Gröbner basis also allows to establish the explicit relationship between the unidentified parameters, which (by eliminating  $T$  from the penultimate two equations) is  $\bar{v} - (\bar{\phi} - 0.8)^2 = 0$ . Any pair of  $\bar{v}$  and  $\bar{\phi}$  meeting this restriction and consistent with the underlying support for deep parameters  $\Theta$  is observationally equivalent to  $v = 1$  and  $\phi = 1.8$ . Naturally, this conclusion perfectly matches that following from the analytical solution given by equation (52), but we arrived at it as if we did not know the latter. It also immediately follows that fixing either  $\bar{v}$  or  $\bar{\phi}$  renders the model globally identified at the considered  $\theta$ .

## 6.2 An-Schorfheide model

### 6.2.1 Model summary

When written in a log-linearized form, the model is given by the following equations

$$x_t = \mathbb{E}_t x_{t+1} + g_t - E_t g_{t+1} - \frac{1}{\tau} (R_t - \mathbb{E}_t \pi_{t+1} - \mathbb{E}_t z_{t+1}) \quad (55)$$

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa (x_t - g_t) \quad (56)$$

$$R_t = \rho_m R_{t-1} + (1 - \rho_m) [\psi_1 \pi_t + \psi_2 (x_t - g_t)] + \varepsilon_{m,t} \quad (57)$$

$$z_t = \rho_z z_{t-1} + \rho_{zg} g_{t-1} + \varepsilon_{z,t} \quad (58)$$

$$g_t = \rho_g g_{t-1} + \rho_{gz} z_{t-1} + \varepsilon_{g,t} \quad (59)$$

There are three endogenous variables in the model: detrended output  $x_t$ , inflation  $\pi_t$  and the interest rate  $R_t$ . They are driven by two exogenous AR(1) processes for productivity growth  $z_t$  and government spending  $g_t$ , with innovations  $\varepsilon_{z,t}$  and  $\varepsilon_{g,t}$ , respectively, and by an i.i.d. monetary policy shock  $\varepsilon_{m,t}$ . All of the i.i.d. innovations are assumed to be mutually uncorrelated and their variances are  $v_z$ ,  $v_g$  and  $v_m$ , respectively. The 13-dimensional vector of deep parameters is hence  $\theta = [ \tau \ \beta \ \kappa \ \psi_1 \ \psi_2 \ \rho_m \ \rho_z \ \rho_{zg} \ \rho_g \ \rho_{gz} \ v_z \ v_g \ v_m ]'$ .

The model can be cast in form (1), with states  $s_t = [ z_t \ g_t \ R_t ]'$ , policy variables  $p_t = [ x_t \ \pi_t ]'$ , shocks  $\varepsilon_t = [ \varepsilon_{z,t} \ \varepsilon_{g,t} \ \varepsilon_{m,t} ]'$  and matrices  $\Gamma_0$ ,  $\Gamma_1$ ,  $\Gamma_2$ ,  $\Gamma_3$  and  $\Sigma$  given by

$$\Gamma_0 = \begin{bmatrix} 0 & -\tau & 1 & \tau & 0 \\ 0 & \kappa & 0 & -\kappa & 1 \\ 0 & (1 - \rho_m)\psi_2 & 1 & -(1 - \rho_m)\psi_2 & -(1 - \rho_m)\psi_1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}; \quad \Gamma_1 = \begin{bmatrix} 1 & -\tau & 0 & \tau & 1 \\ 0 & 0 & 0 & 0 & \beta \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Gamma_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \rho_m \\ \rho_z & \rho_{zg} & 0 \\ \rho_{gz} & \rho_g & 0 \end{bmatrix}; \quad \Gamma_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}; \quad \Sigma = \begin{bmatrix} v_z & 0 & 0 \\ 0 & v_g & 0 \\ 0 & 0 & v_m \end{bmatrix}$$

where we have multiplied equation (55) by  $\tau$  so that all coefficients in the model equilibrium conditions are polynomials. As a result, we do not need to define any auxiliary parameters and can implement our identification analysis directly on the deep model parameters, i.e.  $\alpha = \theta$ .<sup>19</sup> The vector of observable variables is  $y_t = [ R_t \ x_t \ \pi_t ]'$  and there are no measurement errors, which means that  $H = [ 0_{3 \times 2} \ I_3 ]$  and  $J = [ 0_{3 \times 3} ]$ .

### 6.2.2 Global identification failure in a locally identified model

Let us start with the following benchmark parametrization:  $\tau = 2$ ,  $\beta = 0.9975$ ,  $\kappa = 0.33$ ,  $\psi_1 = 1.5$ ,  $\psi_2 = 0.125$ ,  $\rho_m = 0.75$ ,  $\rho_z = 0.9$ ,  $\rho_{zg} = 0.1$ ,  $\rho_g = 0.95$ ,  $\rho_{gz} = -0.075$ ,  $v_z = 0.09$ ,  $v_g = 0.36$ ,  $v_m = 0.04$ . These values are the same as in An and Schorfheide (2007), except for  $\rho_{zg}$  and  $\rho_{gz}$ , which are taken from Kocięcki and Kolasa (2018). Calculating the Gröbner basis results in the following set of solutions for  $\bar{\theta}$ :

<sup>19</sup>More precisely, in the original An-Schorfheide model  $\kappa$  is actually a semi-structural parameter, linked to the deep model parameters via  $\kappa = \tau \frac{1-\nu}{\nu\pi^2\phi}$ . Since  $\nu$ ,  $\pi$  and  $\phi$  do not show up anywhere else in the model equations, including them separately in  $\theta$  instead of combining into  $\kappa$  trivially leads to (local) identification failure.

$$\begin{aligned}
0 &= u^2 - 1.8697u + 0.8697 \\
&\dots \\
\bar{v}_m &= 0.04 \\
\bar{v}_g &= -0.3128 + 0.6728u \\
\bar{v}_z &= 0.1279 - 0.0379u \\
\bar{\rho}_{gz} &= 0.0209 - 0.0959u \\
\bar{\rho}_g &= 0.9345 + 0.0155u \\
\bar{\rho}_{zg} &= 0.2415 - 0.1415u \\
\bar{\rho}_z &= 0.9155 - 0.0155u \\
\bar{\rho}_m &= 0.75 \\
\bar{\psi}_2 &= 0.2516 - 0.1266u \\
\bar{\psi}_1 &= 1.3131 + 0.1869u \\
\bar{\kappa} &= 0.4912 - 0.1612u \\
\bar{\beta} &= 0.5351 + 0.4624u \\
\bar{\tau} &= 2
\end{aligned} \tag{60}$$

where  $u$  is the second element of the second row in matrix  $T$ . To save space, we skip above the equations determining the solutions for other “unknowns” in the system of identification conditions (23)-(31) as they are not needed to arrive at identification conclusions for  $\theta$  – all we need to know is the solution for  $u$  and it is fully determined by the first equation in (60). We also report the numbers rounded to four decimal digits, even though they are in fact arbitrarily accurate numbers.

As we can see, three structural parameters, namely  $\bar{\tau}$ ,  $\bar{\rho}_m$  and  $\bar{v}_m$ , are equal to their respective elements of  $\theta$ , at which we check identification. The remaining elements of  $\bar{\theta}$  are parametrized by  $u$ , which needs to be consistent with the quadratic restriction in the first equation, implying that  $u = 1$  or  $u = 0.8697$ . It is easy to verify that, in the former case, we get our benchmark parameter vector  $\theta$ , while the latter case results in an observationally equivalent model parametrization that is exactly the same as that obtained by Kocięcki and Kolasa (2018) with their numerical algorithm. We have thus a formal and constructive proof that the AS model is locally but not globally identified at  $\theta$ , and that the identification failure concerns all deep parameters but  $\tau$ ,  $\rho_m$  and  $v_m$ . Moreover, looking at the Gröbner basis (60) immediately reveals that fixing any of the unidentified parameters renders the model globally identified.

### 6.2.3 Local identification failure

Let us now consider the same benchmark parameter vector  $\theta$ , except that we rule out any spillovers between productivity and government spending shocks, i.e. fix  $\rho_{zg} = \rho_{gz} = 0$ . Calculating the

Gröbner basis yields:

$$\begin{aligned}
0 &= u^2 - 71.9356u \\
0 &= uw \\
&\dots \\
\bar{v}_m &= 0.04w^2 \\
\bar{v}_g &= 0.36 \\
\bar{v}_z &= 0.09 + 5.5737u \\
\bar{\rho}_g &= 0.95 \\
\bar{\rho}_z &= 0.9 - 0.0054u \\
\bar{\rho}_m &= 0.75w \\
0 &= (\bar{\psi}_2 + 2.7682)u \\
0 &= \bar{\psi}_2(w - 1.3333) - 3.6909 + 3.7326w \\
\bar{\psi}_1 &= 1.5720 - 0.5758\bar{\psi}_2 \\
\bar{\kappa} &= 0.33 \\
\bar{\beta} &= 0.9975 \\
\bar{\tau} &= 2 - 0.6595u
\end{aligned} \tag{61}$$

where, as before, we save space by skipping those elements of the basis that are not necessary for our identification analysis.

Of the two roots of the first equation, only  $u = 0$  does not violate the restrictions on the deep model parameters. In particular, the other root implies  $\tau < 0$ , so we can rule it out. If  $u = 0$ , the second equation does not put any restrictions on  $w$ . Setting  $w = 1$  results in  $\bar{\theta} = \theta$ , any other value of  $w$  meeting the restrictions on the deep parameters gives an alternative parameter vector  $\bar{\theta}$  that is observationally equivalent to  $\theta$ . The identification failure concerns exclusively  $\rho_m$ ,  $\psi_1$ ,  $\psi_2$  and  $v_m$ , which is now proved in a constructive way.

One can think of this failure as local since it applies to any vicinity of  $w = 1$ . This conclusion is consistent with previous papers dealing with this version of the AS model, see e.g. Qu and Tkachenko (2012). Importantly, however, and in contrast to any of the existing approaches to analyze local identification (also Iskrev, 2010; Komunjer and Ng, 2011), our framework analytically produces the whole set of parameter vectors that are observationally equivalent to the one at which we check identification. In this example, the set is one-dimensional and can be written, after some

rearrangement, as follows

$$\begin{aligned}
\bar{\rho}_m &= 0.75w \\
\bar{\psi}_1 &= \frac{-4.2211 + 3.7211w}{w - 1.3333} \\
\bar{\psi}_2 &= \frac{3.6909 - 3.7326w}{w - 1.3333} \\
\bar{v}_m &= 0.04w^2
\end{aligned} \tag{62}$$

where  $w$  is any real number that keeps the alternative model parametrization  $\bar{\theta}$  in the determinacy (and stability) region. A similar (but numerical) concept called nonidentification curves was proposed earlier by Qu and Tkachenko (2012) but it does not generate parametric characterization of the whole set of observationally equivalent parameter values like (62). Having such an explicitly defined set can be useful. For example, one can immediately see from it that our baseline parametrization, which features a positive response of the interest rate to both inflation and output, can be observationally equivalent to one which implies that the central bank's reaction to output is negative (e.g. for  $w = 0.8$ ).

#### 6.2.4 Handling indeterminacy

In the previous two parametrizations of the AS model, we have considered the parameter vectors that imply a unique stable solution. However, our framework can also handle indeterminate cases. To demonstrate it, let us consider the same benchmark  $\theta$  as before, except that now  $\psi_1 = 0.75$ , i.e. half the previously assumed value. It can be easily verified, e.g. by checking the Blanchard-Kahn conditions, that there are infinitely many stable equilibria under such parametrization. As shown by Lubik and Schorfheide (2003), the full set of these equilibria are still given by equations (2) and (3), except that the vector of shocks  $\varepsilon_t$  must include a sufficient number of sunspots. Moreover, expectations of forward-looking variables become new states, and hence need to be included in vector  $s_t$ . As demonstrated by Farmer et al. (2015), an equivalent characterization of indeterminate equilibria is to redefine a subset of expectational errors as new fundamentals. This is what we do in this example.

The order of indeterminacy in the considered model is one, so we need to pick one expectational error. Without loss of generality, let us pick the one associated with the output gap  $x_t$ . Then, the AS model can be written as

$$x_t = \tilde{x}_t + g_t - \mathbb{E}_t g_{t+1} - \frac{1}{\tau}(R_t - \mathbb{E}_t \pi_{t+1} - \mathbb{E}_t z_{t+1}) \tag{63}$$

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa(x_t - g_t) \tag{64}$$

$$R_t = \rho_m R_{t-1} + (1 - \rho_m)[\psi_1 \pi_t + \psi_2(x_t - g_t)] + \varepsilon_{m,t} \tag{65}$$

$$x_t - \tilde{x}_{t-1} = \rho_{sz} \varepsilon_{z,t} + \rho_{sg} \varepsilon_{g,t} + \rho_{sm} \varepsilon_{m,t} + \varepsilon_{s,t} \tag{66}$$

$$z_t = \rho_z z_{t-1} + \rho_{zg} g_{t-1} + \varepsilon_{z,t} \quad (67)$$

$$g_t = \rho_g g_{t-1} + \rho_{gz} z_{t-1} + \varepsilon_{g,t} \quad (68)$$

where  $\tilde{x}_t = \mathbb{E}_t x_{t+1}$ ,  $\varepsilon_{s,t}$  is an i.i.d. sunspot shock with variance  $v_s$  and, as evident from equation (66), we allow for possible correlation between expectational errors and other structural shocks.

As an illustration, we check identification of this model at  $\rho_{sz} = \rho_{sg} = \rho_{sm} = 0.1$ ,  $v_s = 0.01$ . Calculating the Gröbner basis implies a unique solution to our identification restrictions, i.e.  $\bar{\theta} = \theta$ . We have hence proved that the AS model is globally identified at this indeterminate parametrization  $\theta$ . We arrive at the same conclusion also if we fix  $\rho_{zg} = \rho_{gz} = 0$ , thus confirming the outcome obtained by Qu and Tkachenko (2017) with a numerical algorithm that searches over the parameter space.

### 6.3 Smets-Wouters model

We next apply our identification framework to a variant of the widely cited medium-sized DSGE model of Smets and Wouters (2007). The only deviation from the original model is that, following the practice in many policy making institutions, we define the output gap in the monetary policy rule as the deviation of output from its deterministic trend rather than from its potential level. This allows us to leave out the flexible price block and calculate the Gröbner basis in just around 10 seconds. Without this simplification, memory requirements become prohibitively expensive and the Gröbner basis cannot be calculated. Since the model is quite large and its full version well documented in the literature, we describe its structure and all steps in our identification analysis in Appendix A.6, and here we only discuss the conclusions.

By applying our identification framework, we can prove the following results. First, if none of the 41 deep parameters in the model are fixed, the model is not locally (and hence also not globally) identified at the posterior mean reported by Smets and Wouters (2007). Second, after fixing two appropriately selected parameters, which are either the curvature of the Kimball goods market aggregator or the Calvo probability for prices, and either the curvature of the Kimball labor market aggregator or the Calvo probability for wages, the model is globally (and hence also locally) identified.

The part of these findings that concerns local identification hence confirms the results obtained by Iskrev (2010) and Komunjer and Ng (2011), who consider the original version of the model. The only paper that has studied the Smets-Wouters model from a global identification perspective is Qu and Tkachenko (2017). By applying a numerical routine that searches for observationally equivalent parameters, they conclude that the model is globally identified after fixing the five parameters that were originally calibrated in the original paper.<sup>20</sup> The novel finding obtained using our framework is that only two parameters need to be fixed to obtain global identification, at least when the flexible price block is removed.

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<sup>20</sup>The additional three parameters are the depreciation rate, steady-state wage markup and the steady-state share of government purchases in output.

## 6.4 Open economy models

We finally use our framework to study global identification in several variants of open economy models, which has not been done before. Our departure point is the setup developed by Justiniano and Preston (2010), which can be considered a more empirically-oriented version of the small open economy setup by Gali and Monacelli (2005). Similarly to medium-sized closed economy DSGE models in the spirit of Smets and Wouters (2007), the model features imperfect competition, price rigidities, indexation and habits. It also includes two important open economy frictions, namely incomplete international financial markets and local currency pricing in imports. The model is estimated using eight time series, which are home and foreign output, inflation and the short-term interest rates, as well as the terms of trade and real exchange rate.

We additionally consider several extensions to this baseline setup that have been recently emphasized as key to resolving several important puzzles in the open economy literature, see Itskhoki and Mukhin (2021) and Gopinath et al. (2020). These include local currency pricing in exports, strategic complementarities in pricing, and use of imported intermediate inputs in production sold abroad. In Appendix A.7, we present the log-linearized equilibrium conditions of the richest version of the open economy model, and explain how it can be reduced all the way back to the original Justiniano-Preston setup by putting appropriate restrictions on selected parameters. We check global identification at the parameter values calibrated and estimated for Canada (with foreign economy represented by the United States), while the additional parameters showing up in extended versions are set to typical values from the literature. Our analysis takes about half a second for the original (smallest) version of the model but around five minutes for its richest version.

The striking finding of our identification analysis is that, at least when we use the above-mentioned eight time series as observables, all the model variants prove to be globally identified. This is despite we do not fix any of the structural parameters that are usually calibrated rather than estimated when such models are taken to the data.

All of this suggests that, as long as one uses the standard set of observables, observational equivalence might not be the key source of problems encountered while estimating medium-sized open economy DSGE models. A similar conclusion has been recently reached by Adolfson et al. (2019).

## 6.5 General applicability

When successfully executed, our framework can solve the identification problem in a linearized DSGE model, proving its global identification or lack thereof, and offering some analytical characterization of the identification failure in the latter case. However, as we have stressed before, our approach has some limitations that we now briefly summarize. To implement our method, one needs to specify the problem in a particular way, which may demand some analytical effort, especially when the model is large. The necessary steps may involve eliminating redundant states and defining auxiliary parameters such that the problem has a polynomial structure. In sophisticated models, the mapping between the auxiliary and structural parameters may be complex and require numerical



methods to analyze. Another limitation is significant computer power and memory needed to calculate the Gröbner basis. As we have seen from the Smets and Wouters (2007) model example, this may be a binding constraint for some medium-sized DSGE models and standard computing units that are currently in use. Needless to say, this last obstacle will be losing importance if computer performance keeps improving fast.

## 7 Conclusions

In this paper we have developed a comprehensive framework to analyze local and global identification in linearized DSGE models or, more generally, dynamic linear systems with rational or model-consistent expectations. Its main advantage is an analytical flavor, which effectively allows to prove identification or lack thereof. The essence of our approach is application of a Gröbner basis to solve analytically for all roots of a system of polynomial equations, which make up a formal identification condition that we derive.

Calculation of the Gröbner bases is known to be computationally involved for large systems, but we have shown that it can be still successfully applied to small and even some medium-sized DSGE models. One of the conclusions that emerge from the set of studied examples is that ensuring local identification in this class of models usually makes them also globally identified, even though this cannot be always guaranteed. Instead, problems with estimating these models using maximum likelihood, commonly resolved by resorting to Bayesian methods, are much more likely to stem from misspecification or weak identification issues associated with short data series or irregularities in the likelihood function (Al-Sadoon, 2021).

Finally, it is worth stressing that using the concept of a Gröbner basis is not the only possible way to make use of our formal identification condition. One potentially attractive avenue to explore is application of all-solution homotopy methods, recently brought to the attention of economists by Kubler et al. (2014). While numerical in its nature, it may be a useful complement to the Gröbner basis due to its computational advantage, arising from the use of parallelizability.

## Appendix

### A.1 Discussion of Assumption 2

Let us denote  $s = n^2 + nk + rn + rk + \frac{1}{2}k(k+1)$  and  $\mathcal{S} = \{A, B, C, D, \Sigma \in \mathbb{R}^s | \text{rank}(\mathcal{O}) = \text{rank}(\mathcal{C}) = n\} = \mathcal{S}_1 \cap \mathcal{S}_2$ , where  $\mathcal{S}_1 = \{A, B, C, D, \Sigma \in \mathbb{R}^s | \text{rank}(\mathcal{O}) = n\} = \{A, B, C, D, \Sigma \in \mathbb{R}^s | \det(\mathcal{O}'\mathcal{O}) \neq 0\}$ ,  $\mathcal{S}_2 = \{A, B, C, D, \Sigma \in \mathbb{R}^s | \text{rank}(\mathcal{C}) = n\} = \{A, B, C, D, \Sigma \in \mathbb{R}^s | \det(\mathcal{C}\mathcal{C}') \neq 0\}$ . Evidently, both  $\mathcal{S}_1$  and  $\mathcal{S}_2$  are open subsets of  $\mathbb{R}^s$  (being the inverse image of the open set  $\mathbb{R} \setminus \{0\}$ ). Since a finite intersection of open subsets is open, we conclude that  $\mathcal{S}$  is open. Further, since the determinant is a polynomial that is an analytic function of its elements, it implies that  $\mathcal{S}$  is dense in  $\mathbb{R}^s$ . This is because an analytic function such as the determinant cannot be equal to 0 on an open subset of  $\mathbb{R}^s$  unless it is identically equal to zero. Since  $\mathcal{S}$  is an open and dense subset of  $\mathbb{R}^s$ , we conclude that if Assumption 2 is valid for one  $\theta \in \Theta$ , all  $\theta \in \Theta$  such that Assumption 2 is violated form a nowhere dense subset of  $\mathbb{R}^s$  of measure zero.

### A.2 Proof of Theorem 1

Recalling the notation introduced in the main text, let us define the infinite block Hankel matrix as

$$\mathcal{H} = \begin{bmatrix} \Lambda_1 & \Lambda_2 & \Lambda_3 & \cdots \\ \Lambda_2 & \Lambda_3 & \Lambda_4 & \cdots \\ \Lambda_3 & \Lambda_4 & \Lambda_5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \\ \vdots \end{bmatrix} \begin{bmatrix} N & AN & A^2N & A^3N & \cdots \end{bmatrix} \quad (\text{A.1})$$

Assuming stochastic minimality (Assumption 2) and using Sylvester's rank inequality, it may be easily shown that  $\text{rank}(\mathcal{H}) = n$ . Suppose that two sets of deep parameters  $\bar{\theta} \neq \theta$  generate the same autocovariances. Looking at the Hankel matrix, this implies  $\mathcal{O}\mathcal{A}\mathcal{C} = \bar{\mathcal{O}}\bar{\mathcal{A}}\bar{\mathcal{C}}$ . By Assumption 2, it follows that  $\bar{\mathcal{A}} = (\bar{\mathcal{O}}'\bar{\mathcal{O}})^{-1}\bar{\mathcal{O}}'\mathcal{O}\mathcal{A}\mathcal{C}\bar{\mathcal{C}}'(\bar{\mathcal{C}}\bar{\mathcal{C}}')^{-1}$ . Let us denote  $T = (\bar{\mathcal{O}}'\bar{\mathcal{O}})^{-1}\bar{\mathcal{O}}'\mathcal{O}$ , which is nonsingular also by Assumption 2. Since we additionally have  $\mathcal{O}\mathcal{C} = \bar{\mathcal{O}}\bar{\mathcal{C}}$ , we get  $T^{-1} = \mathcal{C}\bar{\mathcal{C}}'(\bar{\mathcal{C}}\bar{\mathcal{C}}')^{-1}$ , hence  $\bar{\mathcal{A}} = T\mathcal{A}T^{-1}$ . Looking at the first block row of the Hankel matrix, we have  $\mathcal{C}\mathcal{C} = \bar{\mathcal{C}}\bar{\mathcal{C}} \Rightarrow \bar{\mathcal{C}} = \mathcal{C}T^{-1}$ . We have hence arrived at the first two conclusions of Theorem 1. Uniqueness of  $T$  follows from equality  $\bar{\mathcal{O}} = \mathcal{O}T^{-1}$  and full column rank of  $\mathcal{O}$ .

Now suppose that  $\bar{\theta} \neq \theta$  results in the same z-spectrum  $\Phi(z) = \bar{\Phi}(z)$  for all  $z \in \mathbb{C}$  in an open annulus containing the unit circle (i.e.  $\bar{\theta} \equiv \theta$ ), that is

$$\begin{aligned} [C(z\mathbf{I}_n - A)^{-1}; \mathbf{I}_r] \begin{bmatrix} B\Sigma B' & B\Sigma D' \\ D\Sigma B' & D\Sigma D' \end{bmatrix} [C(z^{-1}\mathbf{I}_n - A)^{-1}; \mathbf{I}_r]' = \\ [\bar{C}(z\mathbf{I}_n - \bar{A})^{-1}; \mathbf{I}_r] \begin{bmatrix} \bar{B}\bar{\Sigma}\bar{B}' & \bar{B}\bar{\Sigma}\bar{D}' \\ \bar{D}\bar{\Sigma}\bar{B}' & \bar{D}\bar{\Sigma}\bar{D}' \end{bmatrix} [\bar{C}(z^{-1}\mathbf{I}_n - \bar{A})^{-1}; \mathbf{I}_r]' \end{aligned} \quad (\text{A.2})$$

Using  $\bar{A} = TAT^{-1}$  and  $\bar{C} = CT^{-1}$ , we obtain

$$\begin{aligned} & [C(z\mathbf{I}_n - A)^{-1}; \mathbf{I}_r] \begin{bmatrix} B\Sigma B' & B\Sigma D' \\ D\Sigma B' & D\Sigma D' \end{bmatrix} [C(z^{-1}\mathbf{I}_n - A)^{-1}; \mathbf{I}_r]' \\ &= [C(z\mathbf{I}_n - A)^{-1}; \mathbf{I}_r] \begin{bmatrix} T^{-1}\bar{B}\bar{\Sigma}\bar{B}'T'^{-1} & T^{-1}\bar{B}\bar{\Sigma}\bar{D}' \\ \bar{D}\bar{\Sigma}\bar{B}'T'^{-1} & \bar{D}\bar{\Sigma}\bar{D}' \end{bmatrix} [C(z^{-1}\mathbf{I}_n - A)^{-1}; \mathbf{I}_r]' \end{aligned} \quad (\text{A.3})$$

Define the Lyapunov equation evaluated at  $\theta$  as  $P = APA' + B\Sigma B'$ , and that evaluated at  $\bar{\theta}$  as  $\bar{P} = \bar{A}\bar{P}\bar{A}' + \bar{B}\bar{\Sigma}\bar{B}'$ . Using  $\bar{A} = TAT^{-1}$ , the latter may be written as  $T^{-1}\bar{P}T'^{-1} = AT^{-1}\bar{P}T'^{-1}A' + T^{-1}\bar{B}\bar{\Sigma}\bar{B}'T'^{-1}$ . Since  $A$  is stable,  $\bar{P} = T^{-1}\bar{P}T'^{-1}$  is unique.

To proceed further, we need to use a well known lemma, see e.g. Lindquist and Picci (2015), p. 199. Let  $X$  be any symmetric  $n \times n$  matrix  $X$ , then

$$[C(z\mathbf{I}_n - A)^{-1}; \mathbf{I}_r] \begin{bmatrix} X - AXA' & -AXC' \\ -CX A' & -CX C' \end{bmatrix} [C(z^{-1}\mathbf{I}_n - A)^{-1}; \mathbf{I}_r]' = 0 \quad (\text{A.4})$$

Let us use this lemma and subtract (A.4) evaluated at  $X = P$  and at  $X = \bar{P}$  from, respectively, the left and right-hand side of equation (A.3). Keeping in mind the Lyapunov equations, we get

$$[C(z\mathbf{I}_n - A)^{-1}; \mathbf{I}_r] \begin{bmatrix} 0 & S \\ S' & R \end{bmatrix} [C(z^{-1}\mathbf{I}_n - A)^{-1}; \mathbf{I}_r]' = 0 \quad (\text{A.5})$$

where  $S = B\Sigma D' - T^{-1}\bar{B}\bar{\Sigma}\bar{D}' + A(P - \bar{P})C'$  and  $R = D\Sigma D' - \bar{D}\bar{\Sigma}\bar{D}' + C(P - \bar{P})C'$ .

Let  $\rho$  denote the maximum eigenvalue (in modulus) of  $A$  matrix. By Assumption 1,  $0 \leq \rho < 1$ . Hence for all  $|z| > \rho$  we have  $(z\mathbf{I}_n - A)^{-1} = z^{-1}\mathbf{I}_n + z^{-2}A + z^{-3}A^2 + \dots$ , and for all  $|z| < \frac{1}{\rho}$  (if  $\rho = 0$ , set  $\frac{1}{\rho} = \infty$ ) we have  $(z^{-1}\mathbf{I}_n - A)^{-1} = z\mathbf{I}_n + z^2A + z^3A^2 + \dots$ . Multiplying all terms yields

$$S'(z\mathbf{I}_n + z^2A' + z^3A'^2 + \dots)C' + C(z^{-1}\mathbf{I}_n + z^{-2}A + z^{-3}A^2 + \dots)S + R = 0 \quad (\text{A.6})$$

The formula (A.6) is the two-sided z-transform (or Laurent series) of the zero function, which converges absolutely in the open annulus  $\Lambda = \{z \in \mathbb{C} \mid \rho < |z| < \frac{1}{\rho}\}$ . Since  $0 = \sum_{l=-\infty}^{\infty} 0 \cdot z^l$  for all  $z \in \Lambda$ , by uniqueness of the Laurent series it follows that all coefficients of  $z^l$ , for all  $l$ , in (A.6) must be zeros.<sup>21</sup> Using this fact, we get  $R = 0$  and, by stacking together (part of) the remaining restrictions we have  $[C'; A'C'; A'^2C'; \dots; A'^{n-1}C']'S = 0$ . By Assumption 2, the latter yields  $S = B\Sigma D' - T^{-1}\bar{B}\bar{\Sigma}\bar{D}' + A(P - \bar{P})C' = 0$ . Lastly, combining the two Lyapunov equations into one equation we have  $A(P - \bar{P})A' - (P - \bar{P}) = T^{-1}\bar{B}\bar{\Sigma}\bar{B}'T'^{-1} - B\Sigma B'$ . By setting  $Q = P - \bar{P}$ , we arrive at conclusions 3)-5) of Theorem 1. Symmetry and uniqueness of  $Q$  follows from symmetry and uniqueness of  $P$  and  $\bar{P}$ .

The implication in the other direction, which amounts to checking if the spectral density remains

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<sup>21</sup>Putting it other way, note that since  $\{z \in \mathbb{C} \mid |z| = 1\} \subset \Lambda$ , all coefficients in the Laurent series  $f(z) = \sum_{l=-\infty}^{\infty} a_l \cdot z^l$  can be uniquely obtained as  $a_l = \frac{1}{2\pi i} \int_{|z|=1} \frac{f(z)}{z^{l+1}} dz$ , for  $l = 0, \pm 1, \pm 2, \dots$ . Since in our case  $f(z) = 0$ , the result follows.

the same if conclusions 1)-5) in Theorem 1 hold, is easy to demonstrate.<sup>22</sup> Finally, conclusion 6) follows immediately from equality of the first moments of observable variables.

### A.3 Discussion of Assumption 3

We first show that, if  $D\Sigma D'$  is non-singular,<sup>23</sup> then a sufficient condition to make Assumption 3 hold is that matrix  $\Psi = A - B\Sigma D'(D\Sigma D')^{-1}C$  is stable, i.e. its all eigenvalues are strictly less than 1 in modulus. To see it, let us additionally define  $\tilde{M} = B\Sigma B' - B\Sigma D'(D\Sigma D')^{-1}D\Sigma B'$ . Then, by Theorem 5.4. in Katayama (2005), Assumption 3 holds if and only if, for any  $z \in \mathbb{C}$  with  $|z| \geq 1$ , both  $\text{rank}[\Psi - zI_n : \tilde{M}^{\frac{1}{2}}] = n$  and  $\text{rank} \begin{bmatrix} \Psi - zI_n \\ C \end{bmatrix} = n$ . When all eigenvalues of  $\Psi$  are such that  $|z| < 1$ , then for all  $z \in \mathbb{C}$  with  $|z| \geq 1$ ,  $\Psi - zI_n$  is nonsingular, i.e.  $\text{rank}(\Psi - zI_n) = n$ .

Moreover, when  $D$  is nonsingular, so that  $\Psi = A - BD^{-1}C$ , then stability of  $\Psi$  is necessary and sufficient for Assumption 3. To see it, note that, when  $D$  is nonsingular, then  $\tilde{M} = 0$ . Suppose that Assumption 3 holds, but  $\Psi$  is not stable, which means that there is at least one  $|z| \geq 1$  such that  $\Psi - zI_n$  is singular, i.e.  $\text{rank}(\Psi - zI_n) < n$ . In such a case  $\text{rank}[\Psi - zI_n : \tilde{M}^{\frac{1}{2}}] = \text{rank}[\Psi - zI_n : 0] = \text{rank}(\Psi - zI_n) < n$ , for all  $|z| \geq 1$ , cannot hold. Hence we arrive at a contradiction, which implies that  $\Psi$  must be stable.

As a matter of fact, if  $D$  is nonsingular, then stability of  $\Psi$  is equivalent to the ‘‘poor man’s’’ invertibility condition in Fernández-Villaverde et al. (2007) and is almost identical to Assumption 4-S in Komunjer and Ng (2011), i.e. left-invertibility of the transfer function.<sup>24</sup> Moreover, to the extent that Assumption 4-S generalizes the ‘‘poor man’s’’ invertibility condition for the case  $r > k$ , the condition concerning stability of  $\Psi$  can be thought of as generalizing the latter for the case  $r < k$ .

### A.4 Proof of Proposition 1

By definition  $\Lambda_0 - CPC' = D\Sigma D'$ ,  $N - APC' = B\Sigma D'$ ,  $P - APA' = B\Sigma B'$ . Let us write those equations in the form of the following symmetric matrix

$$\begin{bmatrix} P - APA' & N - APC' \\ N' - CPA' & \Lambda_0 - CPC' \end{bmatrix} = \begin{bmatrix} B\Sigma B' & B\Sigma D' \\ D\Sigma B' & D\Sigma D' \end{bmatrix} \quad (\text{A.7})$$

On the other hand, by Assumption 2 we have  $\bar{A} = TAT^{-1}$  and  $\bar{C} = CT^{-1}$ . This implies  $\Lambda_l = CA^{l-1}N = \bar{C}\bar{A}^{l-1}\bar{N} = CA^{l-1}T^{-1}\bar{N}$ , for  $l > 0$ . Using assumption  $\text{rank}(\mathcal{O}) = n$ , we conclude that  $\bar{N} = TN$ . Hence, for all  $\bar{\theta}$ , we can write an analogous symmetric matrix

$$\begin{bmatrix} \tilde{P} - \tilde{A}\tilde{P}A' & N - \tilde{A}\tilde{P}C' \\ N' - \tilde{C}\tilde{P}A' & \Lambda_0 - \tilde{C}\tilde{P}C' \end{bmatrix} = \begin{bmatrix} T^{-1}\bar{B}\bar{\Sigma}\bar{B}'T'^{-1} & T^{-1}\bar{B}\bar{\Sigma}\bar{D}' \\ \bar{D}\bar{\Sigma}\bar{B}'T'^{-1} & \bar{D}\bar{\Sigma}\bar{D}' \end{bmatrix} \quad (\text{A.8})$$

<sup>22</sup>A similar note also applies to the proofs of Propositions 1 and 2 to be presented below.

<sup>23</sup>Non-singularity of  $D\Sigma D'$  is also imposed by Komunjer and Ng (2011) as Assumption 4-NS.

<sup>24</sup>Assumption 4-S in the case of nonsingular  $D$  is equivalent to the statement that all eigenvalues of  $A - BD^{-1}C$  are less than or equal to 1 in modulus.

where  $\tilde{P} = T^{-1}\bar{P}T'^{-1}$ . To proceed, we will need the following lemma

**Lemma 1.** *Let Assumptions 1 and 3 hold. Then  $W = AWA' + K\Sigma_aK'$  has a unique solution with respect to  $W$ . In addition  $P = W + \Pi$ , where  $P = E(s_t s_t')$  and  $\Pi$  is the solution to the Riccati equation (8).*

*Proof.* Since  $\Pi = A\Pi A' + B\Sigma B' - K\Sigma_aK'$  and  $P = APA' + B\Sigma B'$ , we have  $P - \Pi = A(P - \Pi)A' + K\Sigma_aK'$ . Since  $A$  is stable by Assumption 1 and  $K\Sigma_aK'$  is unique (because  $\Pi$  is unique),  $W = P - \Pi$  is also unique.  $\square$

Using Lemma 1, we can rewrite (A.7) and (A.8) as

$$\begin{bmatrix} W - AWA' & N - AWC' \\ N' - CWA' & \Lambda_0 - CWC' \end{bmatrix} = \begin{bmatrix} A\Pi A' - \Pi + B\Sigma B' & A\Pi C' + B\Sigma D' \\ C\Pi A' + D\Sigma B' & C\Pi C' + D\Sigma D' \end{bmatrix} \quad (\text{A.9})$$

and

$$\begin{bmatrix} \tilde{W} - A\tilde{W}A' & N - A\tilde{W}C' \\ N' - C\tilde{W}A' & \Lambda_0 - C\tilde{W}C' \end{bmatrix} = \begin{bmatrix} A\tilde{\Pi}A' - \tilde{\Pi} + T^{-1}\bar{B}\bar{\Sigma}\bar{B}'T'^{-1} & A\tilde{\Pi}C' + T^{-1}\bar{B}\bar{\Sigma}\bar{D}' \\ C\tilde{\Pi}A' + \bar{D}\bar{\Sigma}\bar{B}'T'^{-1} & C\tilde{\Pi}C' + \bar{D}\bar{\Sigma}\bar{D}' \end{bmatrix} \quad (\text{A.10})$$

where  $\tilde{W} = T^{-1}\bar{W}T'^{-1}$  and  $\tilde{\Pi} = T^{-1}\bar{\Pi}T'^{-1}$ . Using (A.9), we can equivalently write the equation for  $W$  as  $W = AWA' + (N - AWC')(\Lambda_0 - CWC')^{-1}(N - AWC)'$ . Evaluating the latter at  $\bar{\theta}$ , after some simple algebra, we can show that  $\tilde{W} = A\tilde{W}A' + (N - A\tilde{W}C')(\Lambda_0 - C\tilde{W}C')^{-1}(N - A\tilde{W}C)'$ . By Lemma 1, we conclude that  $\tilde{W} = W$ .

Hence, the right hand sides of (A.9) and (A.10) are equal, and  $\bar{K} = TK$ ,  $\bar{\Sigma}_a = \Sigma_a$ . Moreover,  $AQ^*A' - Q^* = T^{-1}\bar{B}\bar{\Sigma}\bar{B}'T'^{-1} - B\Sigma B'$ , where  $Q^* = \Pi - \tilde{\Pi}$ . However, since  $P = W + \Pi$ ,  $\tilde{P} = \tilde{W} + \tilde{\Pi}$  and  $\tilde{W} = W$ , we have  $Q^* = \Pi - \tilde{\Pi} = P - \tilde{P} = Q$ . Hence the additional conclusion 3) from Theorem 1 reappears automatically.

## A.5 Proof of Proposition 2

Let us consider the following matrix

$$\begin{bmatrix} \mathbf{I}_n & -V \\ 0 & \mathbf{I}_r \end{bmatrix} \begin{bmatrix} P - APA' & N - APC' \\ N' - CPA' & \Lambda_0 - CPC' \end{bmatrix} \begin{bmatrix} \mathbf{I}_n & 0 \\ -V' & \mathbf{I}_r \end{bmatrix} \quad (\text{A.11})$$

where  $V = (N - APC')(\Lambda_0 - CPC')^{-1}$ . Clearly, using expression (A.7) and the fact that, by Assumption 4,  $\text{rank} \begin{bmatrix} B \\ D \end{bmatrix} = r$ , the inner matrix in (A.11) has rank  $r$ , and the whole matrix (A.11) has also rank  $r$ . Multiplying all terms in (A.11) we have

$$\begin{bmatrix} Z & 0 \\ 0 & \Lambda_0 - CPC' \end{bmatrix} \quad (\text{A.12})$$

where  $Z = P - APA' - (N - APC')(\Lambda_0 - CPC')^{-1}(N - APC)'$ . Using Assumption 4, we have  $\text{rank}(\Lambda_0 - CPC') = \text{rank}(D\Sigma D') = r$  and

$$r = \text{rank} \begin{bmatrix} P - APA' & N - APC' \\ N' - CPA' & \Lambda_0 - CPC' \end{bmatrix} = r + \text{rank}(Z) \quad (\text{A.13})$$

It follows  $Z = 0$  i.e.  $P$  solves the equation for  $W$ . By Lemma 1, we conclude that  $P = W$ , hence  $\Pi = 0$ . Proceeding similarly we can get an analogous result for any other  $\bar{\theta}$ , which leads us to the finding that  $\tilde{P} = W$ . Since  $W = \tilde{W}$  (see the proof of Proposition 1), we get  $\tilde{\Pi} = 0$ .

We conclude that not only the right hand sides of (A.9) and (A.10) are equal, but we can also put  $\Pi = \tilde{\Pi} = 0$ . As a result, we arrive at

$$\begin{bmatrix} B\Sigma B' & B\Sigma D' \\ D\Sigma B' & D\Sigma D' \end{bmatrix} = \begin{bmatrix} T^{-1}\bar{B}\bar{\Sigma}\bar{B}'T'^{-1} & T^{-1}\bar{B}\bar{\Sigma}\bar{D}' \\ \bar{D}\bar{\Sigma}\bar{B}'T'^{-1} & \bar{D}\bar{\Sigma}\bar{D}' \end{bmatrix} \quad (\text{A.14})$$

From (A.14), it follows quite easily that  $\bar{B} = TBU$ ,  $\bar{D} = DU$  and  $\bar{\Sigma} = U^{-1}\Sigma U'^{-1}$ , for some (unique) nonsingular  $k \times k$  matrix  $U$ .

## A.6 Identification of the Smets-Wouters model

The considered model is made of the following 24 equations:

$$y_t = \alpha_1 i_t + (1 - \alpha_1 - g_y)c_t + \varpi z_t + \varepsilon_t^g \quad (\text{A.15})$$

$$c_t = \alpha_2 c_{t-1} + (1 - \alpha_2)\mathbb{E}_t c_{t+1} + \alpha_3(l_t - \mathbb{E}_t l_{t+1}) - \alpha_4(r_t - \mathbb{E}_t \pi_{t+1} + \varepsilon_t^b) \quad (\text{A.16})$$

$$i_t = \alpha_5 i_{t-1} + (1 - \alpha_5)\mathbb{E}_t i_{t+1} + \alpha_6 q_t + \varepsilon_t^i \quad (\text{A.17})$$

$$q_t = \alpha_7 \mathbb{E}_t q_{t+1} + (1 - \alpha_7)\mathbb{E}_t r_{t+1}^k - (r_t - \mathbb{E}_t \pi_{t+1} + \varepsilon_t^b) \quad (\text{A.18})$$

$$y_t = \phi_p [\varpi k_t^s + (1 - \varpi)l_t + \varepsilon_t^a] \quad (\text{A.19})$$

$$k_t^s = k_{t-1} + z_t \quad (\text{A.20})$$

$$\psi z_t = (1 - \psi)r_t^k \quad (\text{A.21})$$

$$r_t^k = -(k_t^s - l_t) + w_t \quad (\text{A.22})$$

$$k_t = \alpha_8 k_{t-1} + (1 - \alpha_8)i_t + \alpha_9 \varepsilon_t^i \quad (\text{A.23})$$

$$\mu_t^p = \varpi(k_t^s - l_t) + \varepsilon_t^a - w_t \quad (\text{A.24})$$

$$\pi_t = \alpha_{10}\pi_{t-1} + \alpha_{11}\mathbb{E}_t \pi_{t+1} - \alpha_{12}\mu_t^p + \varepsilon_t^p \quad (\text{A.25})$$

$$\mu_t^w = w_t - \sigma_l l_t - \alpha_{13}c_t + (\alpha_{13} - 1)c_{t-1} \quad (\text{A.26})$$

$$w_t = \alpha_5 w_{t-1} + \alpha_{14}\pi_{t-1} + (1 - \alpha_5)\mathbb{E}_t(w_{t+1} + \pi_{t+1}) - \alpha_{15}\pi_t - \alpha_{16}\mu_t^w + \varepsilon_t^w \quad (\text{A.27})$$

$$\tilde{r}_t = \rho r_t - r_{\Delta y} y_t \quad (\text{A.28})$$

$$r_t = \tilde{r}_{t-1} + (1 - \rho)r_\pi \pi_t + [(1 - \rho)r_y + r_{\Delta y}]y_t + \varepsilon_t^r \quad (\text{A.29})$$

$$\varepsilon_t^a = \rho_a \varepsilon_{t-1}^a + \eta_t^a \quad (\text{A.30})$$

$$\varepsilon_t^b = \rho_b \varepsilon_{t-1}^b + \eta_t^b \quad (\text{A.31})$$

$$\varepsilon_t^g = \rho_g \varepsilon_{t-1}^g + \rho_{ga} \eta_t^a + \eta_t^g \quad (\text{A.32})$$

$$\varepsilon_t^i = \rho_i \varepsilon_{t-1}^i + \eta_t^i \quad (\text{A.33})$$

$$\varepsilon_t^r = \rho_r \varepsilon_{t-1}^r + \eta_t^r \quad (\text{A.34})$$

$$\varepsilon_t^p = \tilde{\varepsilon}_{t-1}^p + \eta_t^p \quad (\text{A.35})$$

$$\tilde{\varepsilon}_t^p = \rho_p \tilde{\varepsilon}_{t-1}^p + (\rho_p - \mu_p) \eta_t^p \quad (\text{A.36})$$

$$\varepsilon_t^w = \tilde{\varepsilon}_{t-1}^w + \eta_t^w \quad (\text{A.37})$$

$$\tilde{\varepsilon}_t^w = \rho_w \tilde{\varepsilon}_{t-1}^w + (\rho_w - \mu_w) \eta_t^w \quad (\text{A.38})$$

and the covariance matrix of shocks, denoted by  $\eta_t$  with appropriate superscripts, is  $\Sigma = \text{diag}([v_a v_b v_g v_i v_r v_p v_w])$ .

Compared to the original paper by Smets and Wouters (2007), we define the output gap in the monetary policy feedback rule as the deviation of output from its deterministic trend rather than from its hypothetical level in the absence of nominal rigidities and markup shocks. Additionally, to fit the model to our identification framework, and in particular to meet Assumption 2, we use a state-space representation of the ARMA processes for markup shocks and rewrite the monetary policy rule using an appropriately defined auxiliary variable. See Supplement to Komunjer and Ng (2011) and Kocięcki and Kolasa (2018) for more details.

There are 41 deep parameters in the model, i.e.  $\theta = [\gamma \ \pi \ l \ \beta \ \delta \ g_y \ \sigma_c \ \lambda \ \varphi \ \phi_p \ \varpi \ \psi \ \iota_p \ \xi_p \ \varepsilon_p \ \sigma_l \ \phi_w \ \iota_w \ \xi_w \ \varepsilon_w \ \rho \ r_\pi \ r_y \ r_{\Delta y} \ \rho_a \ \rho_b \ \rho_g \ \rho_r \ \rho_r \ \rho_p \ \rho_w \ \rho_{ga} \ \mu_p \ \mu_w \ v_a \ v_b \ v_g \ v_r \ v_r \ v_p \ v_w]$ .<sup>25</sup> While writing the model equations above using the semi-structural parameters, we have defined the following objects:  $\alpha_1 = \frac{(\gamma-1+\delta)\varpi}{\beta^{-1}\gamma^{\sigma_c}-1+\delta}$ ,  $\alpha_2 = \frac{\lambda\gamma^{-1}}{1+\lambda\gamma^{-1}}$ ,  $\alpha_3 = \frac{(1-\varpi)(\sigma_c-1)}{\phi_w\sigma_c(1+\lambda\gamma^{-1})(1-\alpha_1-g_y)}$ ,  $\alpha_4 = \frac{1-\lambda\gamma^{-1}}{(1+\lambda\gamma^{-1})\sigma_c}$ ,  $\alpha_5 = \frac{1}{1+\beta\gamma^{1-\sigma_c}}$ ,  $\alpha_6 = \frac{1}{(1+\beta\gamma^{1-\sigma_c})\varphi\gamma^2}$ ,  $\alpha_7 = \beta\gamma^{-\sigma_c}(1-\delta)$ ,  $\alpha_8 = (1-\delta)\gamma^{-1}$ ,  $\alpha_9 = (1-\alpha_8)(1+\beta\gamma^{1-\sigma_c})\varphi\gamma^2$ ,  $\alpha_{10} = \frac{\iota_p}{1+\beta\gamma^{1-\sigma_c}\iota_p}$ ,  $\alpha_{11} = \frac{\beta\gamma^{1-\sigma_c}}{1+\beta\gamma^{1-\sigma_c}\iota_p}$ ,  $\alpha_{12} = \frac{(1-\beta\gamma^{1-\sigma_c}\xi_p)(1-\xi_p)}{(1+\beta\gamma^{1-\sigma_c}\iota_p)\xi_p[(\phi_p-1)\varepsilon_p+1]}$ ,  $\alpha_{13} = \frac{1}{1-\lambda\gamma^{-1}}$ ,  $\alpha_{14} = \frac{\iota_w}{1+\beta\gamma^{1-\sigma_c}}$ ,  $\alpha_{15} = \frac{1+\beta\gamma^{1-\sigma_c}\iota_w}{1+\beta\gamma^{1-\sigma_c}}$ ,  $\alpha_{16} = \frac{(1-\beta\gamma^{1-\sigma_c}\xi_w)(1-\xi_w)}{(1+\beta\gamma^{1-\sigma_c})\xi_w[(\phi_w-1)\varepsilon_w+1]}$ . Applying them substitutes the following elements of  $\theta$ :  $\beta$ ,  $\delta$ ,  $\sigma_c$ ,  $\lambda$ ,  $\varphi$ ,  $\iota_p$ ,  $\xi_p$ ,  $\varepsilon_p$ ,  $\phi_w$ ,  $\iota_w$ ,  $\xi_w$ ,  $\varepsilon_w$ . The vector of semi-structural parameters is then  $\alpha = [\alpha_1 \ \dots \ \alpha_{16} \ \gamma \ \pi \ l \ g_y \ \phi_p \ \varpi \ \psi \ \sigma_l \ \rho \ r_\pi \ r_y \ r_{\Delta y} \ \rho_a \ \rho_b \ \rho_g \ \rho_r \ \rho_r \ \rho_p \ \rho_w \ \rho_{ga} \ \mu_p \ \mu_w \ v_a \ v_b \ v_g \ v_r \ v_r \ v_p \ v_w]'$ , and hence has four elements more than  $\theta$ . This is because we do not take into account all cross-equation restrictions implied by the model's deep parameters while defining  $\alpha$ . This means that, if our identification

<sup>25</sup>The notation follows exactly Smets and Wouters (2007), except that (i) we replace  $\alpha$  with  $\varpi$  as the former is already reserved in our paper to denote the vector of semi-structural parameters, (ii) we denote the steady state levels of inflation and labor simply as  $\pi$  and  $l$ , i.e. without bars, as these we use to indicate the observationally equivalent alternative parameter values, and (iii) we use variance  $v$  rather than standard deviation  $\sigma$  to measure shock volatility as the latter are obviously identified only up to a sign.

conditions generate any  $\bar{\alpha} \neq \alpha$ , we will need to check if it is consistent with some  $\bar{\theta} \in \Theta$ . Since the Smets-Wouters model is estimated using non-demeaned data, we can use first moments to our identification analysis, associated with restriction  $\bar{M} = M$  in our identification conditions derived in Section 4. This implies in particular  $\bar{\gamma} = \gamma$ ,  $\bar{\pi} = \pi$  and  $\bar{l} = l$ , as these parameters determine average growth in output, inflation and labor, respectively, all of which are observable. Since these three parameters are identified from the first moments, we can treat them as fixed while deriving the identification conditions.

Calculating the Gröbner basis associated with this identification problem at the posterior mean reported in Smets and Wouters (2007), and for the seven observable variables that they use, reveals that the only solution to the system (23)-(31) is such that  $\bar{\alpha} = \alpha$ , which formally proves that all semi-structural parameters that show up in equations (A.15)-(A.38) and in the shock covariance matrix  $\Sigma$  are globally identified. To formulate the conclusions about the deep parameters, one needs to examine the analytical links between  $\alpha$  and  $\theta$  listed above. This is relatively straightforward, especially if we take into account that  $\gamma$  is identified from the first moments. In particular, for given  $\gamma$ ,  $\alpha_2$  uniquely determines  $\lambda$ , then  $\alpha_4$  pins down  $\sigma_c$ ,  $\alpha_7$  determines  $\delta$ , while  $\alpha_5$  pins down  $\beta$ . This further allows to obtain uniquely  $\varphi$  from  $\alpha_6$ ,  $\phi_w$  from  $\alpha_3$ ,  $\iota_p$  from  $\alpha_{11}$ , and  $\iota_w$  from  $\alpha_{14}$ . Out of the remaining four deep parameters,  $\xi_p$  and  $\varepsilon_p$  are linked only to  $\alpha_{12}$  while  $\xi_w$  and  $\varepsilon_w$  show up only in the definition of  $\alpha_{16}$ , and hence they are not identified. To achieve identification, one needs to fix one parameter in each of these two pairs.

## A.7 Identification of open economy models

The richest version of the considered open economy model is given by the following 22 equations:

$$(1+h)\alpha_1 c_t = \alpha_1 E_t c_{t+1} + h\alpha_1 c_{t-1} + (g_t - E_t g_{t+1}) - (i_t - E_t \pi_{t+1}) \quad (\text{A.39})$$

$$(1-\varpi)[(1-\omega)c_t + \omega x_t] = y_t - (1-\varpi)\eta\varpi s_t + \varpi\lambda p_{H,t}^* - \varpi y_t^* \quad (\text{A.40})$$

$$s_t - s_{t-1} = \pi_{F,t} - \pi_{H,t} \quad (\text{A.41})$$

$$q_t = \psi_{F,t} + (1-\varpi)s_t \quad (\text{A.42})$$

$$\pi_{H,t} - \delta_H \pi_{H,t-1} = \beta(E_t \pi_{H,t+1} - \delta_H \pi_{H,t}) + \alpha_2 [(1-\gamma_H)m c_t + \gamma_H \varpi s_t] \quad (\text{A.43})$$

$$m c_t = \varphi y_t - (1+\varphi)z_t - \varphi \omega x_t + \varpi s_t + (1-\omega)\alpha_1(c_t - h c_{t-1}) \quad (\text{A.44})$$

$$\pi_{F,t} - \delta_F \pi_{F,t-1} = \beta(E_t \pi_{F,t+1} - \delta_F \pi_{F,t}) + \alpha_4 [(1-\gamma_F)\psi_{F,t} - \gamma_F(1-\varpi)s_t] + c p_t \quad (\text{A.45})$$

$$\pi_t = \pi_{H,t} + \varpi(s_t - s_{t-1}) \quad (\text{A.46})$$

$$(i_t - E_t \pi_{t+1}) - (i_t^* - E_t \pi_{t+1}^*) = E_t q_{t+1} - q_t - \chi a_t - \phi_t \quad (\text{A.47})$$

$$\beta a_t = a_{t-1} + \beta[y_t - (1-\omega)c_t - \omega x_t + \varpi p_{H,t}^*] \quad (\text{A.48})$$

$$\tilde{i}_t = \psi_i i_t - (1-\psi_i)\psi_{\Delta y} y_t - (1-\psi_i)\psi_e q_t \quad (\text{A.49})$$



$$i_t = \tilde{i}_{t-1} + (1 - \psi_i)\psi_\pi\pi_t + (1 - \psi_i)(\psi_y + \psi_{\Delta y})y_t + (1 - \psi_i)\psi_e(q_t - \pi_t^* + \pi_t) + \eta_t^m \quad (\text{A.50})$$

$$\pi_{H,t}^* - \delta_H^*\pi_{H,t-1} = \beta(E_t\pi_{H,t+1}^* - \delta_H^*\pi_{H,t}^*) + \alpha_3 [(1 - \gamma_H^*)(mc_t - q_t - \varpi s_t) - p_{H,t}^*] \quad (\text{A.51})$$

$$\pi_{H,t}^* = p_{H,t}^* - p_{H,t-1}^* + \pi_t^* \quad (\text{A.52})$$

$$(1 + \varphi)(y_t - z_t) + (1 - \omega)\alpha_1(c_t - hc_{t-1}) = \omega\varphi x_t \quad (\text{A.53})$$

$$z_t = \rho_z z_{t-1} + \eta_t^z \quad (\text{A.54})$$

$$g_t = \rho_g g_{t-1} + \eta_t^g \quad (\text{A.55})$$

$$cp_t = \rho_{cp} cp_{t-1} + \eta_t^{cp} \quad (\text{A.56})$$

$$\phi_t = \rho_\phi \phi_{t-1} + \eta_t^\phi \quad (\text{A.57})$$

$$\begin{bmatrix} \pi_t^* \\ y_t^* \\ i_t^* \end{bmatrix} = A_V \begin{bmatrix} \pi_{t-1}^* \\ y_{t-1}^* \\ i_{t-1}^* \end{bmatrix} + \begin{bmatrix} \eta_t^{\pi^*} \\ \eta_t^{y^*} \\ \eta_t^{i^*} \end{bmatrix} \quad (\text{A.58})$$

and the covariance matrix of shocks denoted by  $\eta_t$  with appropriate superscripts is

$$\Sigma = \begin{bmatrix} \Sigma_M & 0 \\ 0 & \Sigma_V \end{bmatrix} \quad \Sigma_M = \text{diag}([v_m \ v_z \ v_g \ v_{cp} \ v_\phi]) \quad \Sigma_V = \begin{bmatrix} v_{\pi^*} & v_{\pi^*y^*} & v_{\pi^*i^*} \\ v_{\pi^*y^*} & v_{y^*} & v_{y^*i^*} \\ v_{\pi^*i^*} & v_{y^*i^*} & v_{i^*} \end{bmatrix} \quad (\text{A.59})$$

where  $A_V$  and  $\Sigma_V$  are  $3 \times 3$  matrices with, respectively, foreign VAR coefficients and the covariance structure of VAR innovations. Note that  $A_V$  contains 9 independent elements, while the number of independent elements in  $\Sigma_V$  is 6. As in the case of the Smets-Wouters model described in section A.6, the monetary policy rule is written using an appropriately defined auxiliary variable to meet Assumption 2.

There are 47 deep parameters in the model, i.e.  $\theta = [h \ \sigma \ \varpi \ \omega \ \eta \ \lambda \ \varphi \ \beta \ \delta_H \ \xi_H \ \gamma_H \ \delta_H^* \ \xi_H^* \ \gamma_H^* \ \delta_F \ \xi_F \ \gamma_F \ \chi \ \psi_i \ \psi_\pi \ \psi_y \ \psi_{\Delta y} \ \psi_e \ \rho_z \ \rho_g \ \rho_{cp} \ \rho_\phi \ v_m \ v_z \ v_g \ v_{cp} \ v_\phi \ v_{\pi^*} \ v_{y^*} \ v_{i^*} \ v_{\pi^*y^*} \ v_{\pi^*i^*} \ v_{y^*i^*} \ \text{vec}(A_V)]'$ . While writing the model equations we define  $\alpha_1 = \frac{\sigma}{1-h}$ ,  $\alpha_2 = \frac{(1-\xi_H)(1-\beta\xi_H)}{\xi_H}$ ,  $\alpha_3 = \frac{(1-\xi_H^*)(1-\beta\xi_H^*)}{\xi_H^*}$  and  $\alpha_4 = \frac{(1-\xi_F)(1-\beta\xi_F)}{\xi_F}$ . Using these auxiliary definitions eliminates  $\sigma$ ,  $\xi_H$ ,  $\xi_H^*$  and  $\xi_F$ , respectively, so that the vector of semi-structural parameters is  $\alpha = [h \ \varpi \ \omega \ \eta \ \lambda \ \varphi \ \beta \ \delta_H \ \gamma_H \ \delta_H^* \ \gamma_H^* \ \delta_F \ \gamma_F \ \chi \ \psi_i \ \psi_\pi \ \psi_y \ \psi_{\Delta y} \ \psi_e \ \rho_z \ \rho_g \ \rho_{cp} \ \rho_\phi \ v_m \ v_z \ v_g \ v_{cp} \ v_\phi \ v_{\pi^*} \ v_{y^*} \ v_{i^*} \ v_{\pi^*y^*} \ v_{\pi^*i^*} \ v_{y^*i^*} \ \text{vec}(A_V)]'$ ,  $\alpha_1 \ \alpha_2 \ \alpha_3 \ \alpha_4$ , which also has 47 elements. Note that the mapping from  $\alpha$  to  $\theta$  is straightforward, so by solving the identification problem for the former we immediately obtain the outcome for the latter.

Compared to the original Justiniano-Preston setup, the equilibrium conditions (A.39)-(A.58) feature several additional structural parameters.<sup>26</sup> These are:  $\omega$ , which is the share of intermediate

<sup>26</sup>Otherwise, the notation here follows exactly Justiniano and Preston (2010), except that (i) we replace  $\alpha$  with  $\varpi$  as the former is already reserved in our paper to denote the vector of semi-structural parameters, (ii) we replace the Calvo probabilities  $\theta$  with  $\xi$  as the former we use to denote the vector of deep parameters, (iii) we use variance  $v$

inputs in output,  $\gamma_H, \gamma_H^*, \gamma_F$ , which controls the degree of strategic complementarity in domestic, export and import pricing, and  $\xi_H^*$ , which measures the degree of price stickiness in export sales.

We first analyze global identification for the model described in Justiniano and Preston (2010), which obtains by setting  $\omega = \gamma_H = \gamma_H^* = \gamma_F = 0$  and replacing the Phillips curve for exports (A.51) with the law of one price for domestic production  $-\varpi s_t = p_{H,t}^* + q_t$ . We check identification at the point corresponding to the posterior median estimated by these authors for Canada, see Table I in their paper. We next move to extensions, adding successively the new features until we reach the full version described above. For all of these models, we find that the only solution to our identification conditions (23)-(31) is such that  $\bar{\alpha} = \alpha$ , and hence (after restricting each  $\xi_H, \xi_H^*$  and  $\xi_F$  to lie in the unit interval)  $\bar{\theta} = \theta$ .

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rather than standard deviation *sd* to measure shock volatility, thus avoiding the need to impose an additional sign restriction.

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