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When 0 + ⅓ + ⅓ > ⅔, but 0 + 0 + ⅓ < ⅓. How the median outcome impacts lottery valuation?

Krzysztof Kontek and Michael Birnbaum

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How the median outcome impacts lottery valuation?

Krzysztof Kontek[1] and Michael Birnbaum[2]

Abstract

This paper presents the results of two experiments that exhibit monotonicity violations: some lotteries with three equally likely outcomes are valued more than a superior two-outcome lottery, while others are valued less than an inferior two-outcome lottery. Moreover, the experimental data provide compelling evidence that lottery valuation strongly depends on the value(s) of the middle outcome(s). This contradicts the claim of Cumulative Prospect Theory (CPT) that middle outcomes are assigned lower weights than the extreme ones. Both effects can be observed in the case of four-outcome lotteries. The patterns are persistent for various payoff schedules, and have been observed for subjects from both Poland and California. Incorporating the median outcome value into any modeling of risky decision-making enables these effects to be explained. This paper demonstrates that a simple weighted Expected Utility - Median model describes data involving two- three-, and four-outcome lotteries more accurately than CPT. Moreover, it offers an alternative explanation of "overweighting" of small probabilities, and "underweighting" of large ones – phenomena postulated by CPT.

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Key words: decision – making under risk; monotonicity violations; Expected Utility Theory (EUT); Cumulative Prospect Theory; median.

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1. Introduction

Imagine you have to determine the certainty equivalents (CE) of two lotteries: ($0, ⅓; $100, ⅔) and ($0, ⅓; $100, ⅓; $100, ⅓). The two lotteries are obviously equivalent. The only difference is that the probability of winning $100 is split into two equal halves in the latter. Normatively, the two lotteries should have the same CE. However, as shown in this paper, the latter is valued about 15% more than the former when presented separately in an experiment. The reverse pattern is also observed: a lottery ($0, ⅓; $0, ⅓; $100, ⅓) is valued about 15% less than a lottery ($0, ⅔; $100, ⅓). In this case, splitting the probability of winning $0 decreases the lottery valuation. This indicates monotonicity violations.

Monotonicity is a crucial assumption for most theories of decision-making under risk. Violations, however, have been reported in the literature, e.g. by Birnbaum and Beeghley (1997) in valuations of three-outcome lotteries, and Birnbaum and Veira (1998) in valuations of four-outcome lotteries. In the first paper, the authors considered lotteries with three equally likely outcomes (x, y, z) and manipulated x. In their set up, most x outcomes assumed a value less than y or above z; this changed the range of lottery outcomes each time. The authors observed that a lottery ($4, $39, $45) is valued more than ($4, $12, $96), but after replacing an outcome of $4 by $136, a lottery ($39, $45, $136) is valued less than ($12, $96, $136). This indicates a violation of branch independence. A similar set up, albeit for four-outcome lotteries, was used by Birnbaum and Veira (1998).

Other examples of monotonicity violations in choices between lotteries have been reported by Birnbaum (2008). For instance, he stated that most subjects prefer a lottery ($96, 0.85; $90, 0.05; $12, 0.10) over ($96, 0.90; $14, 0.05; $12, 0.05), despite the former being stochastically dominated by the latter.[3] These experiments served Birnbaum to motivate his configural

[3] To see this, observe that ($96, 0.85; $90, 0.05) is worse than ($96, 0.90), and ($12, 0.10) is worse than ($14, 0.05; $12, 0.05).

weighting and TAX models. His experimental work on monotonicity violations has not, however, found support.

This paper demonstrates other examples of monotonicity violations and shows that they are closely tied with another effect, i.e. that the lottery valuation strongly depends on the value of middle outcomes. This contradicts the claim of Cumulative Prospect Theory (CPT, Tversky and Kahneman, 1992), which argues an inverse S-shaped probability weighting function that assigns middle outcomes lower weights than the extreme ones. This claim, however, can no longer hold. There is strong experimental evidence that middle outcomes are assigned greater weights than extreme ones. This conclusion has already been stated by Birnbaum (1997, 1998, 2008). This calls CPT into question: the probability weighting function derived using binary lotteries takes an inverse S-shape, while that derived using multi-outcome lotteries takes an S-shape.

This paper postulates that the median outcome value should be incorporated into any model of decision-making under risk. A simple weighted Expected Utility - Median model, according to which people tend to shift their CE valuations towards the median outcome value (the mid-point in the case of binary lotteries), is here proposed. As shown, this model not only explains monotonicity violations, but also describes lottery valuations more accurately than CPT in terms of the sum of squared errors and other measures. Importantly, it is consistent: a single weight value assigned to the median can satisfactorily explain the data for two, three-, and four outcome lotteries. In addition, it offers an alternative explanation of "overweighting" of small probabilities, and "underweighting" of large ones.[4]

This paper first presents the idea behind the experiments conducted here (Section 2). It is gen-

[4] The impact of the median outcome may also serve to explain the violations presented by Birnbaum. Observe that a lottery ($4, $39, $45) is valued more than ($4, $12, $96) and the median outcome of $39 is greater than the median outcome of $12; a lottery ($39, $45, $136) is valued less than ($12, $96, $136) and the median outcome of $45 is less than the median outcome of $96; a lottery ($96, 0.85; $90, 0.05, $12, 0.10) is preferred over ($96, 0.90; $14, 0.05, $12, 0.05) and the median outcome of $90 is greater than the median outcome of $14. This topic, however, is beyond the scope of this paper and requires a separate treatment.

erally based on Birnbaum's method of manipulating one or two outcome values while holding the others constant. There are, however, differences in approach. First, in the present experiments, only the middle outcome(s) are manipulated; this way, the range of lottery outcomes remains unchanged, which helps determine the impact of middle outcome(s) on lottery valuation. Second, the experiments involve concurrently two-, three-, and four-outcome lotteries; this allows valuations obtained for lotteries with different numbers of outcomes to be compared. The paper next describes Experiment 1, which was conducted with subjects from Poland (Section 3), and Experiment 2, conducted with subjects from California and Poland (Section 4). Section 5 presents estimation results of a few models of decision-making under risk, including EUT and CPT, as well as weighted ones that involve the median outcome value. The results are discussed in Section 6. Appendixes 1 and 2 detail the instructions used in the two experiments. Appendixes 3 and 4 include the aggregated CE values obtained in the experiments. Individual data are available on request.

2. Experiment – the idea

The idea behind the experiments that were conducted runs as follows. Consider a lottery with three equally likely outcomes such that $x_{min} \leq x \leq x_{max}$. Note that when $x = x_{min}$, the total probability of winning x_{min} is ⅔ and the probability of winning x_{max} is ⅓. This lottery is therefore equivalent (at least from the normative viewpoint) to the binary lottery (x_{min}, ⅔; x_{max}, ⅓). On the other hand, when $x = x_{max}$, the total probability of winning x_{max} is ⅔, and the probability of winning x_{min} is ⅓. This lottery is therefore equivalent to the binary lottery (x_{min}, ⅓; x_{max}, ⅔). It follows that for each x in the range (x_{min}, x_{max}), the CE of the three-outcome lottery (x_{min}, ⅓; x, ⅓; x_{max}, ⅓) should assume a value between the CEs of the stated binary lotteries. In the experiment, the CEs of lotteries (x_{min}, ⅔; x_{max}, ⅓) and (x_{min}, ⅓; x_{max}, ⅔) are first determined. The CEs of the lotteries (x_{min}, ⅓; x, ⅓; x_{max}, ⅓) are then determined for various x values in the range [x_{min}, x_{max}]. Finally, the CEs of the three-outcome lotteries are checked against their predicted values.

The same idea is applied to four-outcome lotteries. Consider a lottery with four equally likely outcomes such that $x_{min} \leq x_2 \leq x_3 \leq x_{max}$. When $x_2 = x_3 = x_{min}$, the total probability of winning x_{min} is ¾ and the probability of winning x_{max} is ¼. This lottery is therefore equivalent to the binary lottery (x_{min}, ¾; x_{max}, ¼). When $x_2 = x_3 = x_{max}$, the total probability of winning x_{max} is ¾, and the probability of winning x_{min} is ¼. This lottery is therefore equivalent to a binary lottery (x_{min}, ¼; x_{max}, ¾). In the experiment, the CEs of the lotteries (x_{min}, ¾; x_{max}, ¼) and (x_{min}, ¼; x_{max}, ¾) are first determined. Then the CEs of the lotteries (x_{min}, ¼; x_2, ¼; x_3, ¼; x_{max}, ¼) are determined for various x_2 and x_3 values in the range [x_{min}, x_{max}].

3. Experiment 1

3.1. Detailed design

Two payoff schedules were used with x_{max} = 300 zł (Schedule 1) and x_{max} = 900 zł[5] (Schedule 2). The lowest outcome, x_{min}, assumed a value of 0 in both schedules. In the case of three-outcome lotteries, the outcome x assumed values of 0, 15, 30, 75, 150, 225, 270, 285, and 300 zł in Schedule 1, and 0, 45, 90, 225, 450, 675, 810, 855, and 900 zł in Schedule 2. This resulted in a total of 18 lotteries. In the case of four outcome lotteries, outcomes x_2 and x_3 (where $x_2 \leq x_3$) assumed values of 0, 30, 150, 270, and 300 zł in Schedule 1, and 0, 90, 450, 810, and 900 zł in Schedule 2. Lotteries (0, ¼; 100, ¼; 200, ¼; 300, ¼) and (0, ¼; 300, ¼; 600, ¼; 900, ¼) were added to the set in Schedule 1 and 2, respectively. This resulted in 16 lotteries for each schedule.

The CEs of the binary lotteries were determined for the following probabilities p of winning x_{max}: 0.01, 0.05, 0.10, 0.25, 0.5, 0.75, 0.9, 0.95, and 0.99. The CE values so obtained were then used to estimate the required values for p = ¼, ⅓, ⅔, and ¾.

[5] Złoty is the Polish currency, \$1 ≈ 4 zł, although the purchasing power for basic goods is closer to parity.

3.2. Participants

The experiment involved 110 subjects - undergraduate students of economics at the Warsaw School of Economics. The age of the participants ranged from 18 to 25 years with a mean of 20.5 years and 52% were women. The students received information about the experiment from their supervisors, who worked with one of the authors of this paper, and agreed to promote the experiment. Participation was voluntary. The participants were given a 12-zł voucher that they could redeem at the campus cafeteria. Subjects were further incentivized by performance. They were informed before the experiment that some of them would be taking part in a real lottery. Four subjects were selected after the data was collected. The two who gave the lowest CE for a particular lottery received the amounts they quoted. The other two were required to take part in a lottery using real money.

As the experiment was conducted on the Internet, the subjects could respond at their convenience. The participants first registered and familiarized themselves with the instructions online (see Appendix 1). They were then required to solve two sample problems. The time to answer all questions was planned at 40-50 minutes, although the participants were asked to work at their own pace.

3.3. CE determination

The term "certainty equivalent" was not used in the instructions, as it is unknown or difficult to understand for most people. Nor was the term "lottery". Instead, the problems were described in a business-oriented way as risky ventures having 2, 3, or 4 scenarios with various probabilities of occurrence. These scenarios were presented in table form. Example problems are demonstrated in Figure 3.1. In the first example, there were three possible scenarios with outcomes: 0, 75, and 300 zł; each occurring with a probability of 33.3%. Note that in the second example, the scenario with an outcome of 450 zł and a probability of 25% appears twice.

Problem 22

33,3%	33,3%	33,3%		100%	
0 zł	75 zł	300 zł	=		zł

Problem 61

25%	25%	25%	25%		100%	
0 zł	450 zł	450 zł	900 zł	=		zł

Figure 3.1: Example problems involving a three- (top) and a four-outcome (bottom) lottery (note that Polish notation requires a comma to separate digits in 33.3%).

The participants had to state the value that the empty 100% box on the right would need to have to make them indifferent between participating in a risky venture and accepting a sure sum of money.

The experiment was conducted on the Internet using a server located in Poland. Problems were presented to the participants in random order. Six HTML forms with 120 randomly ordered problems were prepared (they included other lotteries not discussed in this paper). A given form was randomly assigned to a participant at the moment of starting the experiment.

3.4. Aggregating the data.

Subjects' responses in experiments involving lottery CEs are usually noisy, skewed and contain a large number of outliers. Moreover, people tend to round their CE valuations to the nearest ten, fifty, or even hundred (e.g. 10, 50, 250, 700 rather than 9, 52, 257, 690). Such rounded CE values may then appear several times in responses of different subjects; the term "tied values" is used in the literature on robust statistics to describe repetitive responses (see e.g. Wilcox, 2011, 2012). Example histograms of CE responses obtained for particular lotteries are presented in Figure 3.2.

Therefore, it is of great importance to choose a proper measure of CE location. The mean value is known to be very sensitive to outliers (e.g. the upper left graph in Figure 3.2, which has a mean value of 65.6, but a median value of only 27.5). The median value is less sensitive to outliers. However, it is sensitive to tied values (e.g. middle and right graphs with a median of 200, 300, 250, and 350).

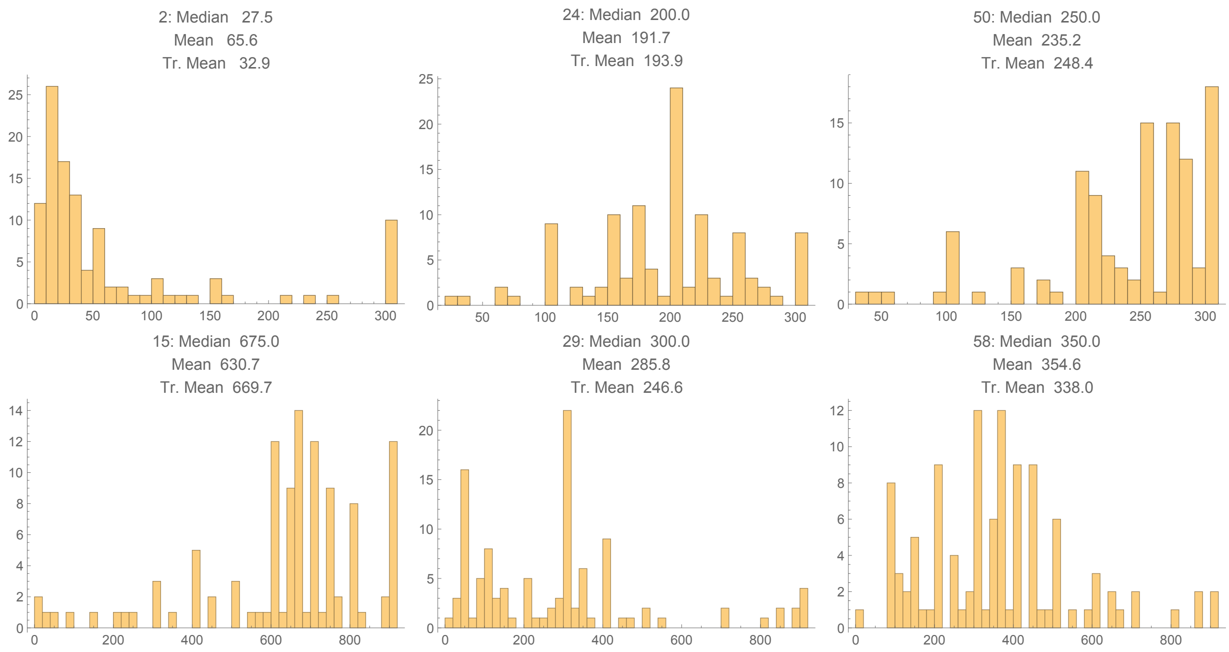

Figure 3.2: Example histograms of CE responses for particular lotteries presented with mean, median and 20% trimmed mean values. First column: binary lotteries; second column: three-outcome lotteries; third column: four-outcome lotteries. First row: Schedule 1; second row: Schedule 2.

Many robust location estimators have been proposed in the literature. The trimmed mean estimator is simple to compute, yet, according to Wilcox (2012), performs often better than more complex ones when sampling from heavy-tailed distributions. The trimmed mean is the mean of the elements in list after dropping a fraction *f* of the smallest and largest elements. Wilcox (2011) suggests *f* = 20% for data in social and behavioral sciences. The aggregated CE values thereby obtained are detailed in Appendix 3.

3.5. Binary lotteries

The aggregated *CE* values for binary lotteries are presented in Figure 3.3 as a function of probability *p* (orange dots). The CE values have been used to estimate a model $CE(p) = x_{max}\, w(p)$ with the help of the two-parameter Prelec (1998) function: $w(p) = Exp[-\gamma(-\ln p)^{\delta}]$. This is the CPT model with a linear value function and a probability weighting function described by the Prelec function.

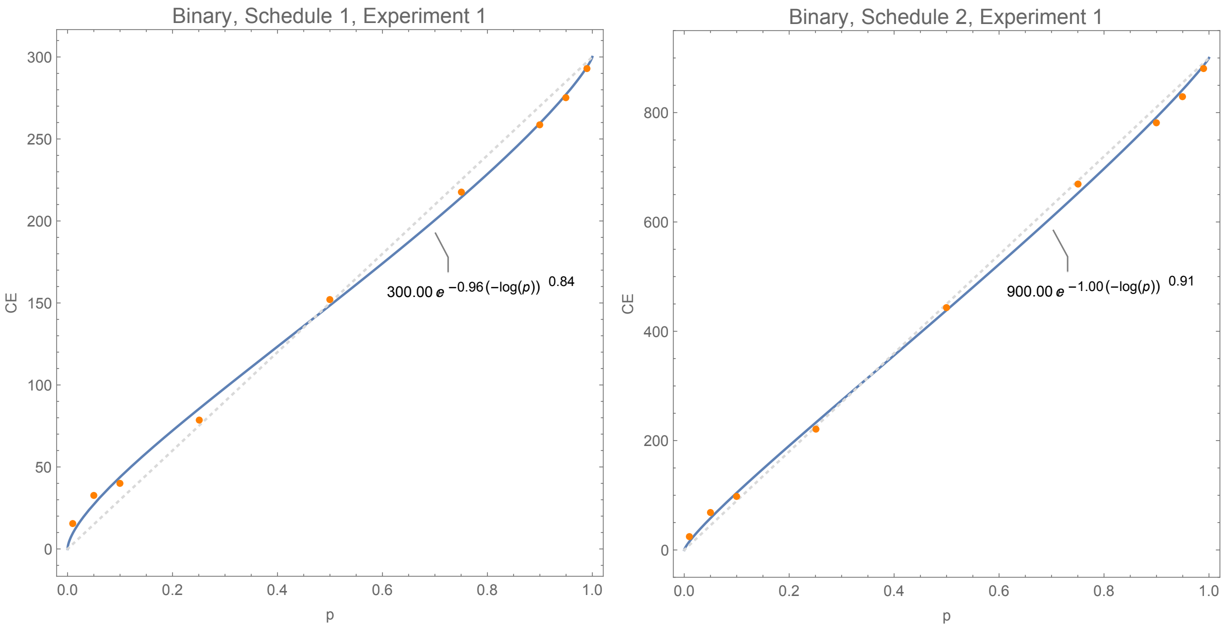

Figure 3.3: CEs of binary lotteries presented as a function of the probability *p* of winning the greater payoff; Schedule 1 (left), and Schedule 2 (right).

As seen, both estimated *CE*(*p*) functions assume an inverse S-shape, usually obtained in experiments involving binary lotteries (e.g. Tversky and Kahneman, 1992). The estimated CE values are therefore greater than the lottery expected value for p = ⅓ and p = ¼, and lower than the lottery expected value for p = ⅔ and p = ¾. CPT explains this phenomenon as overweighting of small probabilities, and underweighting of large ones.

3.6. Three-outcome lotteries.

The aggregated CE values for three-outcome lotteries are presented in Figure 3.4 as a function of the middle outcome x. The two horizontal dotted lines mark the *CE* values for binary lotteries and probabilities ⅓ and ⅔. These values were estimated using binary lotteries (106.6 and 191.6 for Schedule 1, and 301.1 and 579.5 for Schedule 2). The lines define a band in which the *CEs* of three-outcome lotteries should be located for all x values. However, as seen in both graphs, three *CE* values are located below the *CE*(⅓) line, and four *CE* values are located above the *CE*(⅔) line. Only 2 of the 9 *CE* values are within the band supposed by the results obtained for binary lotteries.

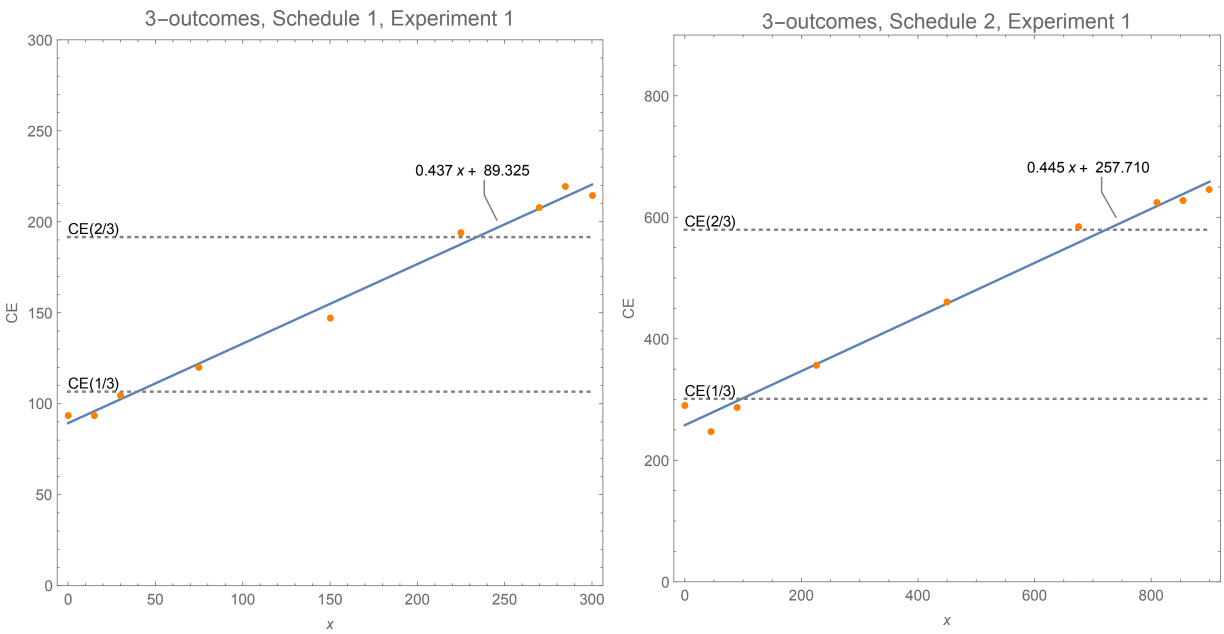

Figure 3.4: *CE* values of three-outcome lotteries presented as a function of *x* for two payoff schedules.

The aggregated *CE* values were used to estimate linear models (see the graphs; the adjusted R-squared values are 0.99 and 0.98 respectively). It follows from the models that the CE under- and overvaluation at both ends of the outcome range is in the order of 15% compared to the *CE*(⅓) and *CE*(⅔) values estimated using binary lotteries. Importantly, the slopes of the two curves (i.e. 0.437 and 0.445) are greater than the probability of the *x* outcome (i.e. ⅓). This indicates that the middle outcome has a big impact on the lottery valuation.

This observation is problematic for the CPT model, as it assigns lower weights to middle outcomes than to extreme ones. In the case considered, CPT assigns weights of 0.355, 0.283, and 0.361 to three equally likely outcomes in Schedule 1 (assuming an inverse S-shaped probability weighting function estimated using binary lotteries), and weights of 0.335, 0.309, and 0.356 in Schedule 2. The pattern observed in Figure 3.4 can therefore only be explained by an S-shaped probability weighting function. This, however, contradicts the shape derived using binary lotteries in this experiment (see Figure 3.3), as well as in other experiments reported in the literature (e.g. Tversky and Kahneman, 1992).

3.7. Four-outcome lotteries.

The aggregated *CE* values for four-outcome lotteries are visualized as 3D surfaces in Figure 3.5 as functions of middle outcomes x_2 and x_3. Two planes are additionally presented. These assume values of *CE*(¼) and *CE*(¾), estimated using binary lotteries (85.3 and 214.3 for Schedule 1, and 232.4 and 652.4 for Schedule 2). As seen, the bottom-left corner of the CE surface is located below the *CE*(¼) plane, whereas the upper-right corner is above the *CE*(¾) plane.

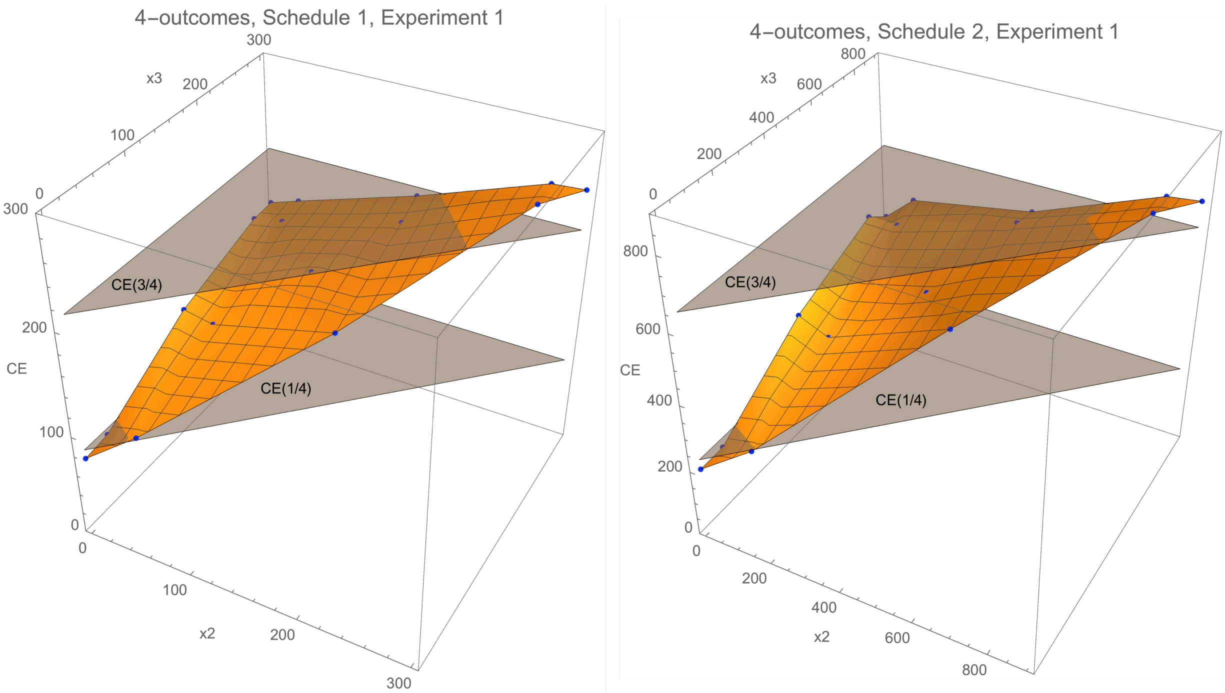

Figure 3.5: *CE* surfaces as a function of x_2 and x_3 for Schedule 1 (left) and Schedule 2 (right). The dots on the surface mark the lotteries involved in the experiment.

The *CE* values have been used to estimate linear models: $CE = 73.5 + 0.304\, x_2 + 0.299\, x_3$ for Schedule 1, and $CE = 192.5 + 0.292\, x_2 + 0.314\, x_3$ for Schedule 2 (the adjusted R-squared values are 0.99 and 0.98 respectively). It follows from the models that the under- and overvaluation of *CE* in both corners are again in the order of 15%. As in the case of three-outcome lotteries, the estimated slope values (0.304, 0.299, 0.292, and 0.314) are greater than the probability of the middle outcomes x_2 and x_3 (i.e. ¼). This stands in disagreement with the inverse S-shaped probability weighting function postulated by CPT and obtained for binary lotteries in the present experiment.

Note that the coefficients for x_2 and x_3 are close to each other. This suggests that the mean of the middle outcomes would describe the data equally well. Figure 3.6 presents CEs of four-outcome lotteries as a function of the mean of x_2 and x_3.

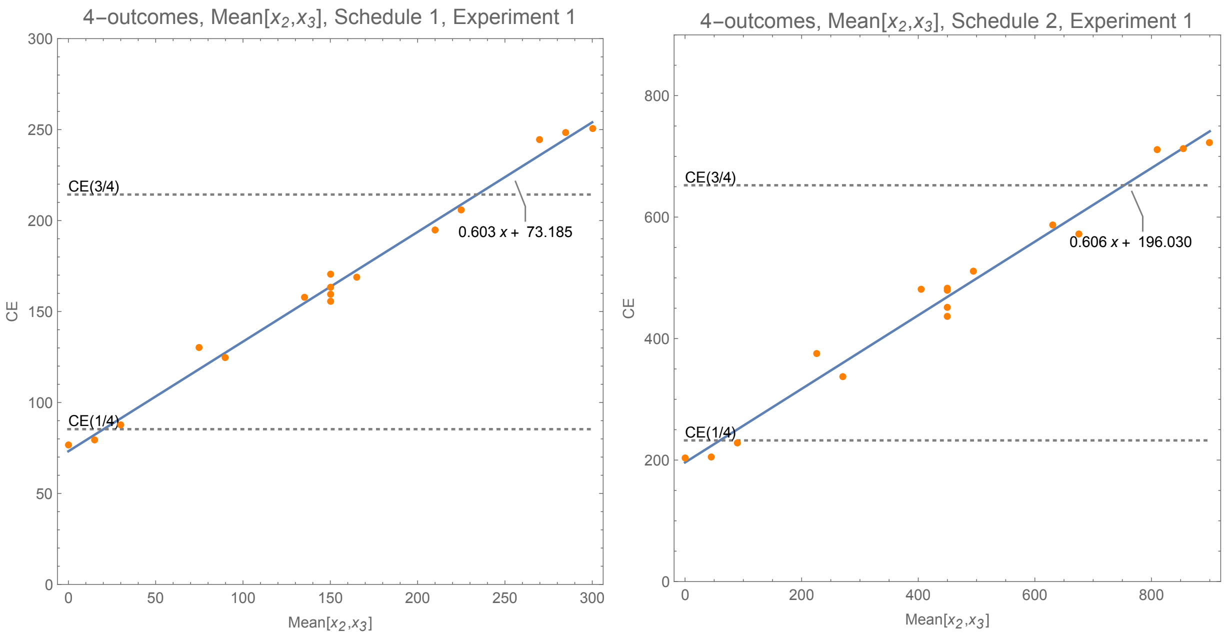

Figure 3.6: *CE* values of four-outcome lotteries presented as a function of the mean of x_2 and x_3 for Schedule 1 (left) and Schedule 2 (right).

As seen, the CEs depend almost linearly on the mean of x_2 and x_3 (the adjusted R-squared values are 0.99 and 0.98 respectively). The relationship becomes only slightly flatter at both ends of the outcome range. The estimated slopes of the two curves (0.603 and 0.606) are greater than the total probability of outcomes x_2 and x_3 (i.e. ½).

3.8. Summary

The patterns obtained for three- and four-outcome lotteries clearly violate monotonicity: some lotteries with three or four equally likely outcomes are valued more than a superior two-outcome lottery, and some less than an inferior two-outcome lottery. Models of decision-making under risk that assume monotonicity (e.g. EUT and CPT) therefore fail to explain these violations. Moreover, estimated slope values for middle outcome(s) are greater than their probabilities of occurrence. This forces the probability weighting function in the CPT model to assume an S-shape, i.e. the very opposite to the inverse S-shape estimated for binary lotteries (more on the model estimation in Section 5). Interestingly, the two problems seem to be interlinked: starting points outside the band require high slope values in the linear models, or alternatively, high slope values lead to valuations outside of the band and to monotonicity violations. This suggests that the two problems have a common origin.

Note that the outcome x in a three-outcome lottery and the mean of x_2 and x_3 in a four-outcome lottery are median lottery outcome values. Moreover, the CE valuations are almost linearly related to these median values. This suggests that the median outcome value plays an important role in the lottery valuation and should therefore be included in any modeling of risky decision-making. Proposals for such an approach are presented in Section 5.

4. Experiment 2

4.1. Detailed design

In Experiment 2, only one payoff schedule was used with x_{min} = \$5 and x_{max} = \$95 (NB: the outcomes were given in \$ in this experiment). In the case of both three- and four-outcome lotteries, the middle outcome(s) assumed a value of \$10, \$20, \$30, \$40, \$50, \$60, \$70, \$80, or \$90. Note that in this setup, the middle outcome(s) never assume(s) the x_{min} or x_{max} values. Each of the three-outcome lotteries was presented to the subjects twice. This resulted in 18 problems to be solved. In the case of four-outcome lotteries, the restriction $x_2 \leq x_3$ did not apply (as in Experiment 1), so the subjects were presented with both e.g. (\$5, \$20, \$60, \$95) and (\$5, \$60, \$20, \$95), but with one (\$5, \$20, \$20, \$95) only. This factorial design leads to 81 lotteries. A lottery (\$5, \$45, \$55, \$95) was added to the set. This resulted in a total of 100 problems to be solved. Binary lotteries were not examined in Experiment 2.

4.2. Participants

Seventy-six subjects took part in the experiment. Sixty-two of them were undergraduate psychology students at the California State University, Fullerton, and 65% of that group were women. Fourteen were undergraduate economics students at the Warsaw School of Economics, and 57% of that group were women. The students received information about the experiment from their supervisors. Participation was voluntary.

The experiment was conducted on the Internet using a server located in the USA (Experiment 2 was prepared by the second author). Thus the subjects could respond at their convenience. Moreover, subjects from California and Poland could participate at the same time. The participants first registered and familiarized themselves with the instructions online (see Appendix 2). They were then required to solve four sample problems. The participants were asked to answer questions at their own pace.

4.3. CE determination

The lotteries were presented in list form. Example problems are demonstrated below:

8. ($5, $70, $95) -

9. ($5, $30, $10, $95) -

10. ($5, $60, $80, $95) -

The participants had to state the value that would make them indifferent between participating in a lottery and accepting a sure sum of money. Problems were presented to the participants in random order.

4.4. Three-outcome lotteries.

The individual CE data have been aggregated using a 20%-trimmed-mean (see Appendix 4 for the detailed results). The aggregated CE values for three-outcome lotteries are presented in

Figure 4.1 as a function of the middle outcome *x*.

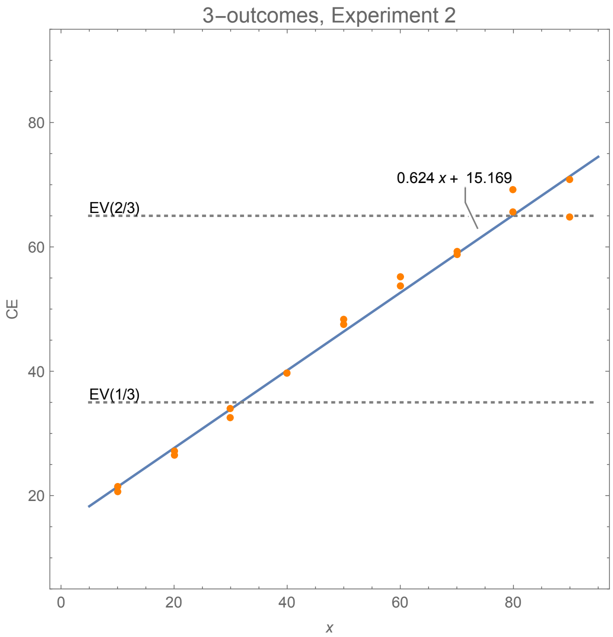

Figure 4.1: Aggregated *CE* values for three-outcome lotteries presented as a function of *x*.

The two horizontal dotted lines mark the *CE* values \$35 and \$65. These are the binary lottery (\$5, 1-*p*; \$95, *p*) expected values for probabilities *p* of ⅓ and ⅔. A linear probability weighting function is assumed here because binary lotteries were not examined in Experiment 2. If an inverse S-shaped probability weighting function had been used, the band of supposed CE values would have been narrower.

As seen, aggregated *CE* values are located below the EV(⅓) line for $x \leq \$30$, and above or on the EV(⅔) line for $x \geq \$80$. Note that two CE points are given for each *x* value, as each lottery was presented to the participants twice. Thus, the average values for $x \geq \$80$ are greater than the lottery expected value for $p = ⅔$.

The *CE* values have been used to estimate a linear model (see the graph; the adjusted R-squared value is 0.98). It follows from the model that CEs of \$18.5 (for $x = \$5$) and \$74.5 (for $x = \$95$) are 48% less, and 15% greater than the respective lottery expected values. The curve slope value of 0.624 is much greater than the probability of the middle outcome *x* (i.e. ⅓). This illustrates the importance of the middle outcome in lottery valuation to even a greater extent than in Experiment 1, where slopes assumed values of 0.437 in Schedule 1 and 0.445 in Schedule 2. Again, this is problematic for the CPT model, as it requires an S-shaped probability weighting function to accommodate such data.

4.5. Four-outcome lotteries.

The aggregated *CE* values for four-outcome lotteries are visualized in Figure 4.2 as a function of middle outcomes x_2 and x_3.

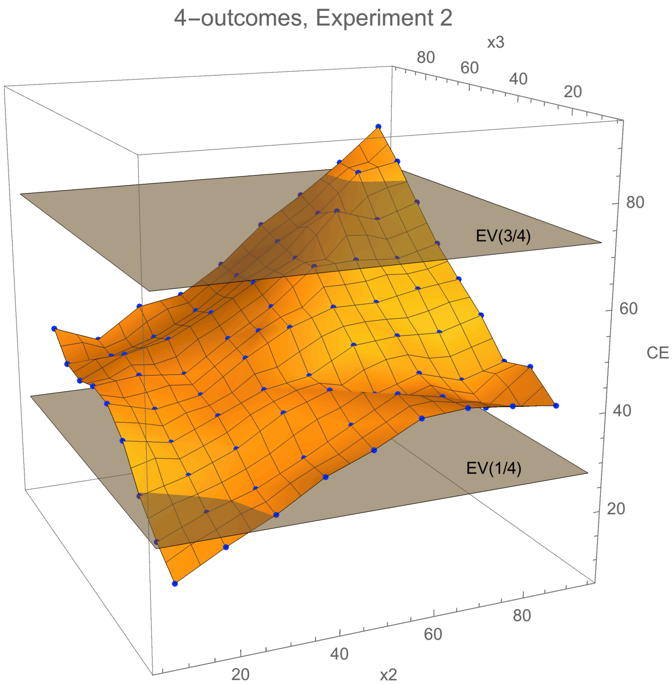

Figure 4.2: *CE* surface as a function of x_2 and x_3 for four-outcome lotteries in Experiment 2. The dots on the surface mark the lotteries involved in the experiment.

Two planes of constant values of \$27.5 (binary lottery expected value for p =¼) and \$72.5 (binary lottery expected value for p =¾) are additionally presented. As seen, the bottom-left corner of the CE surface is located below the EV(¼) plane, whereas the upper-right corner is above the EV(¾) plane.

The *CE* values were used to estimate the linear model: $CE = 13.7 + 0.333\ x_2 + 0.315\ x_3$. The adjusted R-squared value is 0.917, which indicates that the model is not as accurate as in Experiment 1. It follows from the model that the CE is \$16.9 for $x_2 = x_3 =$ \$5, which is 38.5% less than the lottery expected value, whereas the CE is \$75.3 for $x_2 = x_3 =$ \$95, which is 4% more than the lottery expected value. The estimated slope values for the outcomes x_2 and x_3 (i.e. 0.333 and 0.315) are both greater than their probability of occurrence (i.e. ¼).

The CEs of four-outcome lotteries are presented as a function of the mean of x_2 and x_3 in Figure 4.3.

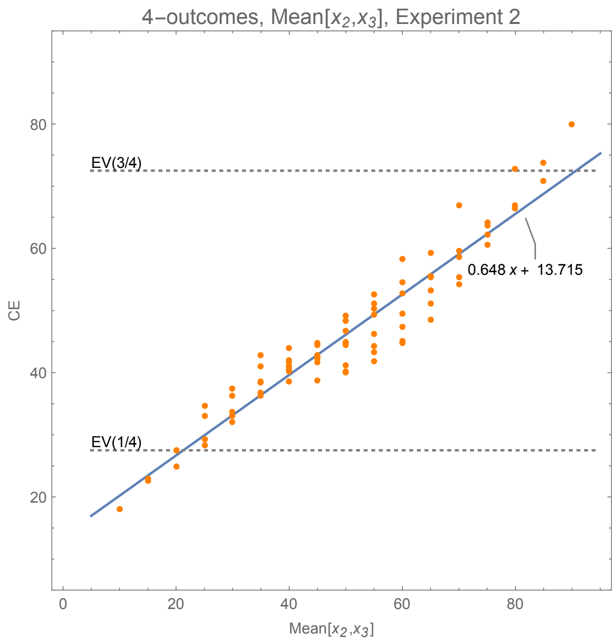

Figure 4.3: *CE* values of four-outcome lotteries presented as a function of the mean of x_2 and x_3.

As seen, CEs strongly depend on the mean of middle outcomes. The linear model has an adjusted R-squared value of 0.917. The estimated slope value of 0.648 is greater than the total probability of the outcomes x_2 and x_3 (i.e. ½), similarly as in Experiment 1.

4.6. Analysis for groups

As the relationship presented in Figure 4.3 is not perfectly linear, separate analyses were performed for the subjects from California and the subjects from Poland. Figure 4.4 shows the dependence of CE values on the middle outcome x for three-outcome lotteries. The pattern stated earlier in this paper is present for both groups.

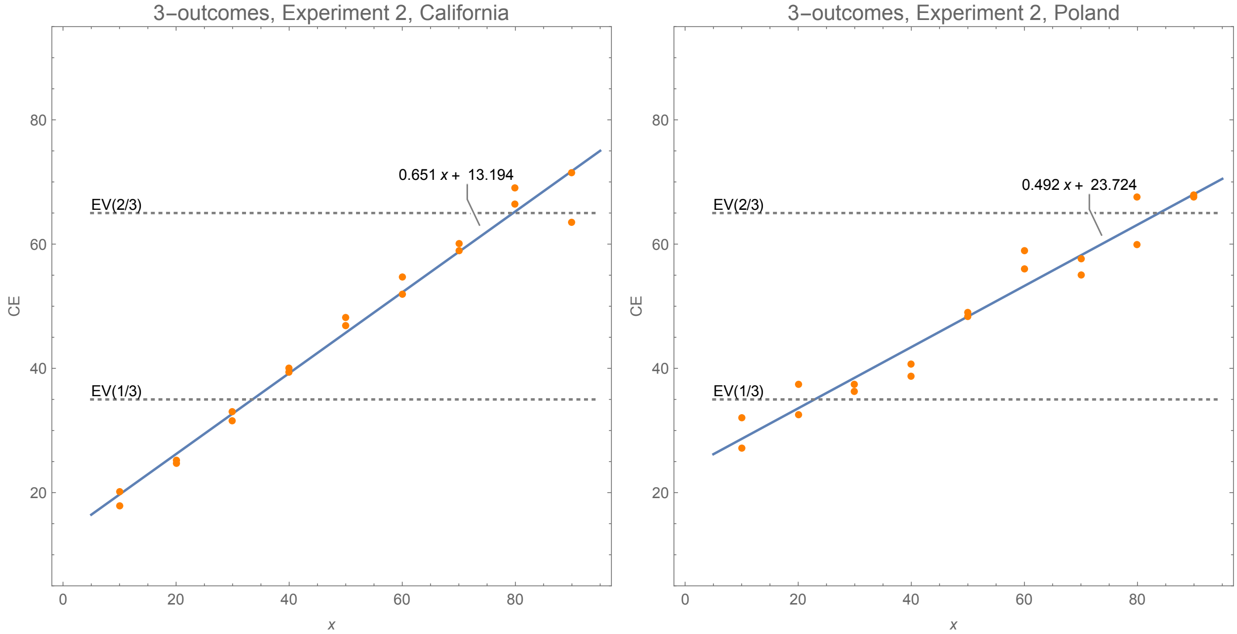

Figure 4.4: *CE* values of three-outcome lotteries presented as a function of *x* for subjects from California (left), and subjects from Poland (right).

However, the slope of the curve for the subjects from California is steeper. In both cases, the slope values (0.651 and 0.492) are much greater than the probability of the middle outcome (i.e. ⅓). CE surfaces for four-outcome lotteries are presented in Figure 4.5.

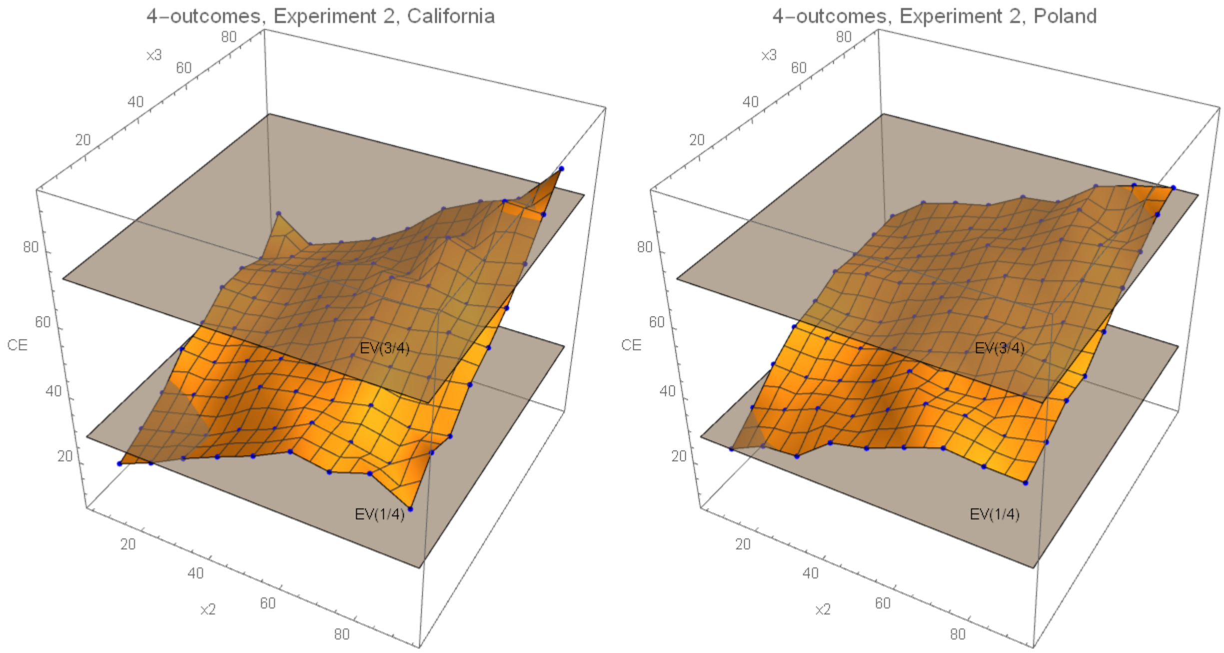

Figure 4.5: *CE* surfaces as a function of x_2 and x_3 for four-outcome lotteries in Experiment 2 for subjects from California (left) and Poland (right). The dots on the surface mark the lotteries involved in the experiment.

As in the case of three-outcome lotteries, the pattern is also present, but less visible for the subjects from Poland. This is confirmed in Figure 4.6, which shows the dependence of CE on the mean of the x_2 and x_3 outcomes.

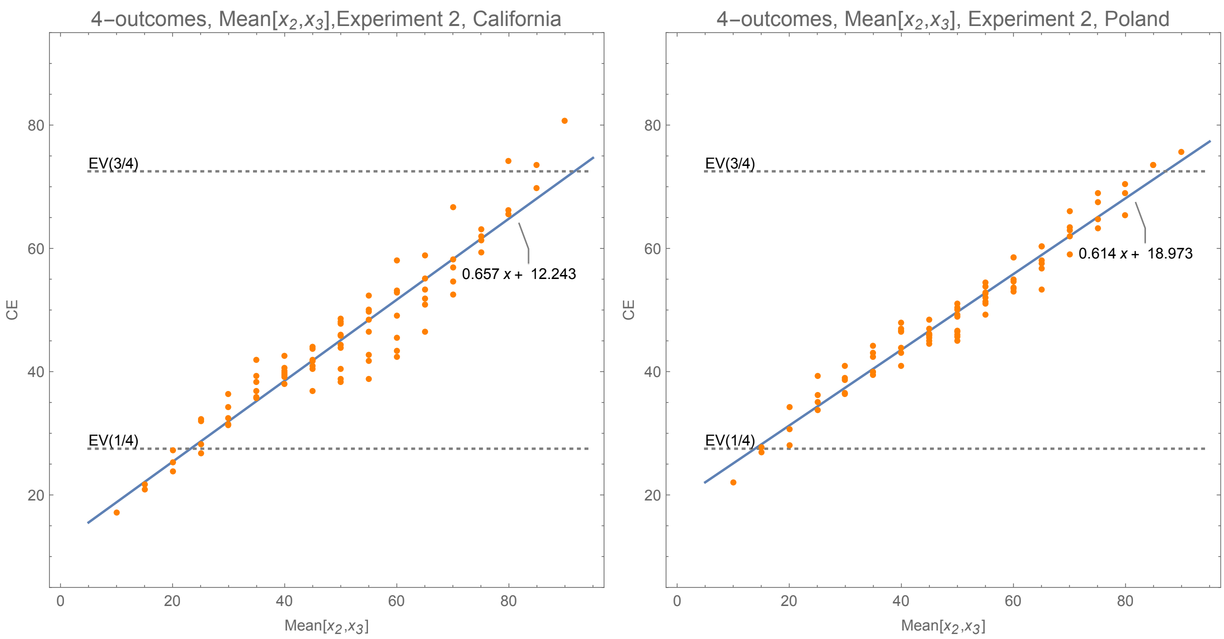

Figure 4.6: *CE* values of four-outcome lotteries presented as a function of the mean of x_2 and x_3 for subjects from California (left) and from Poland (right).

The relationship is closer to linear and the curve is flatter for the Polish subjects. Note that the estimated slope values of 0.657 and 0.614 are both greater than the total probability of middle outcomes (i.e. ½).

4.7. Summary

As in Experiment 1, the patterns obtained indicate monotonicity violations: some lotteries with three (or four) equally likely outcomes are valued more than the binary lottery expected value for $p = ⅔$ (or $p = ¾$ respectively), and some are valued less than the binary lottery expected value for $p = ⅓$ (or $p = ¼$ respectively). The estimated slope values are greater than the respective outcome probabilities (and greater than in Experiment 1). These patterns are also present when the data are analyzed separately for the subjects from California and Poland. Polish subjects demonstrated lower sensitivity to middle outcome(s). It should be borne in mind that they were undergraduate economics students, whereas the subjects from California were undergraduate psychology students. The level of mathematical education, rather than the country of origin, may well be the main factor accounting for the differences.

However that may be, the results obtained in Experiment 2 confirm the main observation stated in Experiment 1, viz. that the median outcome value plays an important role in the lottery valuation. Proposals to incorporate this value into the modeling of risky decision-making are presented in Section 5.

5. Model estimation.

The aggregated CE values have been used to verify the way(s) in which models of decision-making under risk describe the patterns observed. In addition to the established models (i.e. EUT and CPT), two simple weighted models involving the median outcome value are considered.

5.1. Models

A power utility function $u(x) = x^{\alpha}$ is assumed for all models under consideration. How the models evaluate CEs is detailed below.

- Expected Value (EV):

$$CE_{EV} = \sum_{i=1}^{n} x_i p_i$$

- The Expected Utility Theory (EUT) model:

$$CE_{EUT} = u^{-1}\left[\sum_{i=1}^{n} u(x_i) p_i\right]$$

- The Cumulative Prospect Theory (CPT) model:

$$CE_{CPT} = u^{-1}\left\{\sum_{i=1}^{n} u(x_i)\left[w\left(\sum_{j=1}^{i} p_j\right) - w\left(\sum_{j=1}^{i-1} p_j\right)\right]\right\}$$

The probability weighting function is described using the two-parameter Prelec (1998) function $w(p) = Exp\left[-\gamma(-\ln p)^{\delta}\right]$.

To verify the impact of the median outcome value, denoted as *Med*, two weighted models were examined:

- The weighted EV-Median model:

$$CE_{wEV-Med} = (1-w)CE_{EV} + w\,Med$$

- The weighted EUT-Median model:

$$CE_{wEUT-Med} = (1-w)CE_{EUT} + w\,Med$$

In the case of the binary lotteries in Experiment 1, the median outcome value was assumed to be the mid-point of the outcome range, i.e. 150 zł in Schedule 1, and 450 zł in Schedule 2.

Finally, two additional weighted models which involve the mean outcome value, denoted as *Mean*, were examined. The purpose of this was to check whether it is in fact the median, and not the mean value, that impacts the lottery valuation.

- The weighted EV-Mean model:

$$CE_{wEV-Mean} = (1-w)CE_{EV} + w\,Mean$$

- The weighted EUT-Mean model:

$$CE_{wEUT-Mean} = (1-w)CE_{EUT} + w\,Mean$$

The last model is essentially the Prospective Reference Theory (Viscusi, 1989) model, the only difference being that it uses the mean value of the outcomes, rather than the mean value of the outcome utilities. The estimation was performed using the Mathematica NonlinearModelFit function, which constructs a nonlinear least-squares model and assumes that errors are independent and normally distributed.

5.2. Estimation results for Experiment 1

The estimation results for Experiment 1 are presented in Table 5-1.

Model	Sum of sq. err.	AIC	BIC	Parameters		
				Est.value	St.err.	p-value
EV	54 738.5	652.0	656.4			
EUT	51 178.8	647.4	651.8	α = 1.04	0.02	8.50×10^{-55}
CPT	36 175.6	627.8	636.7	α = 1.03	0.12	6.33×10^{-13}
				γ = 1.03	0.10	1.40×10^{-15}
				δ = 1.19	0.04	4.89×10^{-42}
wEV-Med	32 874.2	617.3	621.7	w = 0.10	0.02	5.75×10^{-9}
wEUT-Med	29 468.2	611.8	618.5	α = 1.05	0.02	1.11×10^{-58}
				w = 0.10	0.01	1.81×10^{-9}
wEV-Mean	52 298.3	648.9	653.3	w = 0.05	0.03	0.08
wEUT-Mean	49 244.1	646.8	653.4	α = 1.04	0.02	2.34×10^{-53}
				w = 0.04	0.03	0.11

Table 5-1 Model estimations for Experiment 1.

As seen, both the simple weighted models involving the median outcome value (which has a weight of 0.10 in both models) are more accurate than CPT in terms of both the sum of squared errors, and AIC and BIC measures. Most striking, however, is that the simplest, one-parameter weighted EV – Median model performs better than CPT in describing the two- three-, and four-outcome lotteries examined in Experiment 1. Per contra, weighted models involving the mean outcome value only offer a slight improvement over EV and EUT. This result should be expected, as the outcome mean does not add much information to EV or EUT. The CPT and weighted EUT-Median predictions are presented in Figure 5.1 separately for two-, three-, and four-outcome lotteries (in the last case, the predictions are calculated for $x = x_2 = x_3$).

As seen, the differences between the predictions of the two models for three- and four-outcome lotteries are very small. Most striking, however, is the difference in the predictions for binary lotteries.

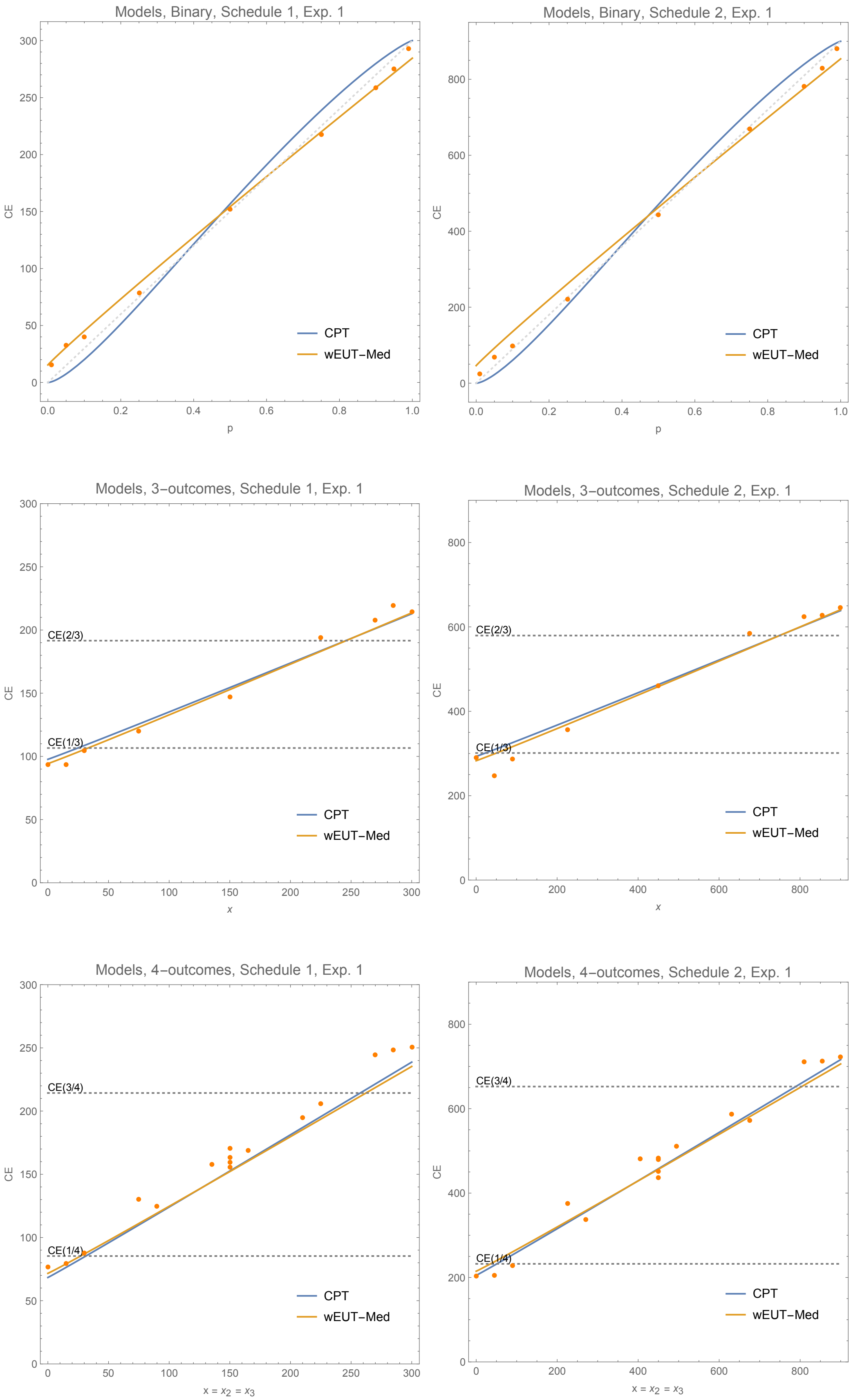

Figure 5.1: CPT and weighted EUT-Median predictions for two-, three-, and four-outcome lotteries, and for the two payoff schedules in Experiment 1.

The weighted EUT-Median model correctly predicts “overweighting” of small probabilities, and “underweighting” of large ones. CPT, on the other hand, predicts the opposite, viz. “underweighting” of small probabilities, and “overweighting” of large ones. This results from the shape of the CPT probability weighting function, estimated for all lotteries, which is S-shaped (see the yellow plots in Figure 5.2). The estimated CPT value and probability weighting functions are presented together with those only obtained for three- and four-outcome lotteries (see the blue curves in Figure 5.2). The latter probability weighting function is even more S-shaped than the one estimated for all lotteries.

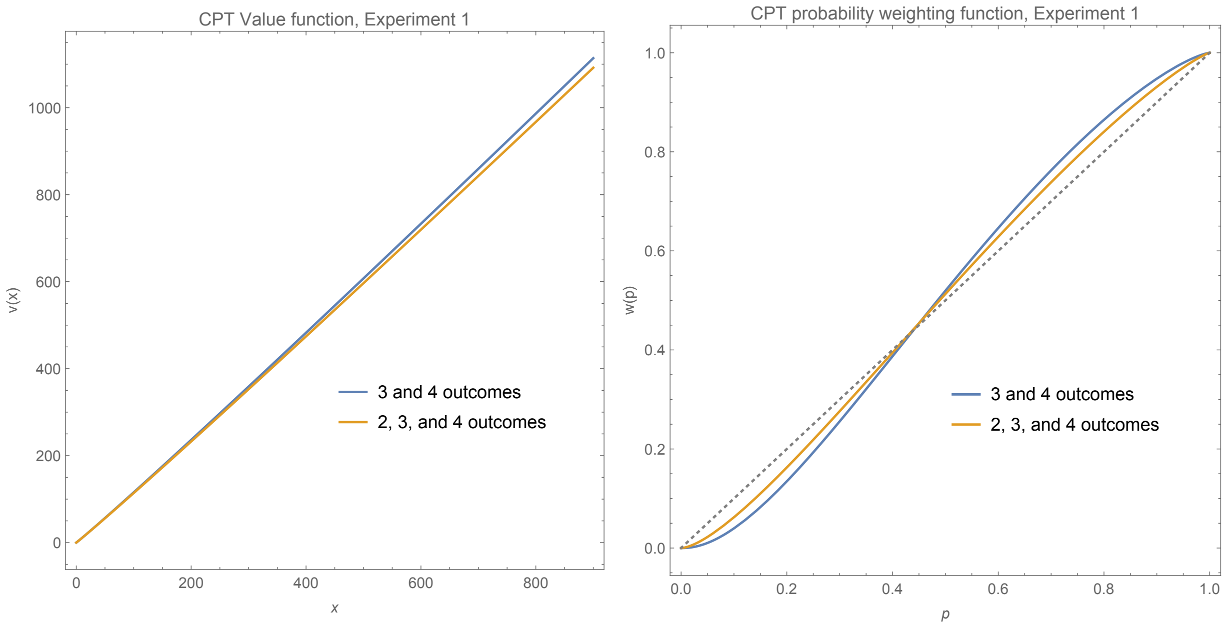

Figure 5.2: CPT value function (left) and probability weighting function (right) estimated using data from Experiment 1 (yellow – all lotteries, blue – only 3- and 4-outcome lotteries).

This demonstrates that three- and four-outcome lotteries require an S-shaped probability weighting function in the CPT model. This conclusion has already been stated on the basis of the slope values in the linear models estimated in Sections 3. Per contra, binary lotteries shall be described using an inverse S-shaped probability weighting function (cf. Figure 3.3).

5.3. Estimation results for Experiment 2.

The estimation results for Experiment 2 are presented in Table 5-2.

Model	Sum of sq. err.	AIC	BIC	Parameters		
				Est.value	St.err.	p-value
EV	4168.29	660.8	666.0			
EUT	2621.91	614.4	619.6	α = 0.78	0.03	1.43×10^{-47}
CPT	1332.65	550.8	561.2	α = 0.87	0.09	6.83×10^{-16}
				γ = 1.27	0.07	6.07×10^{-32}
				δ = 1.51	0.06	5.34×10^{-43}
wEV-Med	2619.7	614.4	619.6	w = 0.36	0.05	1.34×10^{-11}
wEUT-Med	1108.21	530.3	538.1	α = 0.65	0.03	2.40×10^{-36}
				w = 0.36	0.03	1.05×10^{-19}
wEV-Mean	4168.29	660.8	666.0	w = 0.00	2.64×10^{14}	1.00
wEUT-Mean	2621.91	616.4	624.3	α = 0.78	0.80	0.33
				w = 0.00	3.77	1.00

Table 5-2: Model estimations for Experiment 2.

As in Experiment 1, the weighted EUT-Median model is more accurate than CPT in terms of the sum of squared errors, and AIC and BIC measures. The weighted EV-Median model is not that accurate in this case. This is because estimated utility functions indicate a risk aversion attitude (α values from 0.65 to 0.87) that cannot be accommodated by expected value. It should, however, be pointed out that incorporating the median outcome value into the modeling substantially improved the results in the case of EV. Weighted models involving the mean value do not offer any improvement over EV and EUT.[6] The CPT and weighted EUT-Median predictions are presented in Figure 5.3 separately for three- and four-outcome lotteries (in the latter case, the predictions are calculated for $x = x_2 = x_3$).

The differences between the predictions of the two models for four-outcome lotteries are very small. However, the CPT model is far less accurate than the weighted EUT-Median model in the case of three-outcome lotteries.

[6] This results from only using lotteries with equally likely outcomes in this estimation, as this leads to model equivalence.

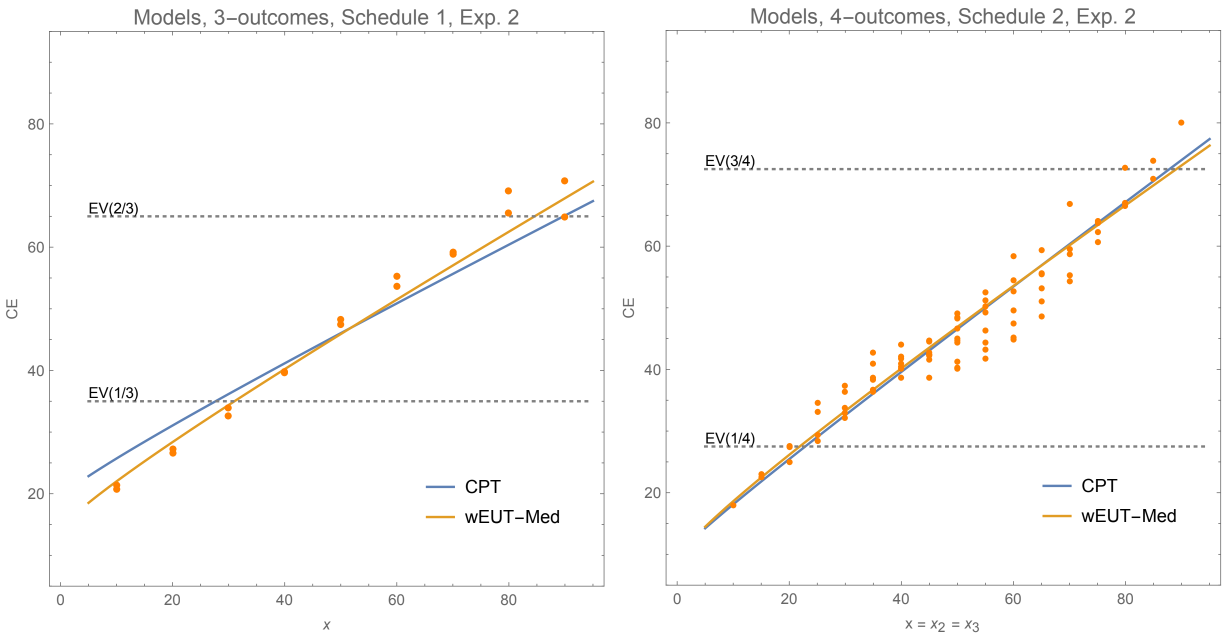

Figure 5.3: CPT and weighted EUT-Median predictions for three- (left) and four-outcome (right) lotteries.

The estimated CPT probability weighting function is even more S-shaped than in Experiment 1 (see Figure 5.4). This was predictable given the higher slope values in the linear models than those in Experiment 1.

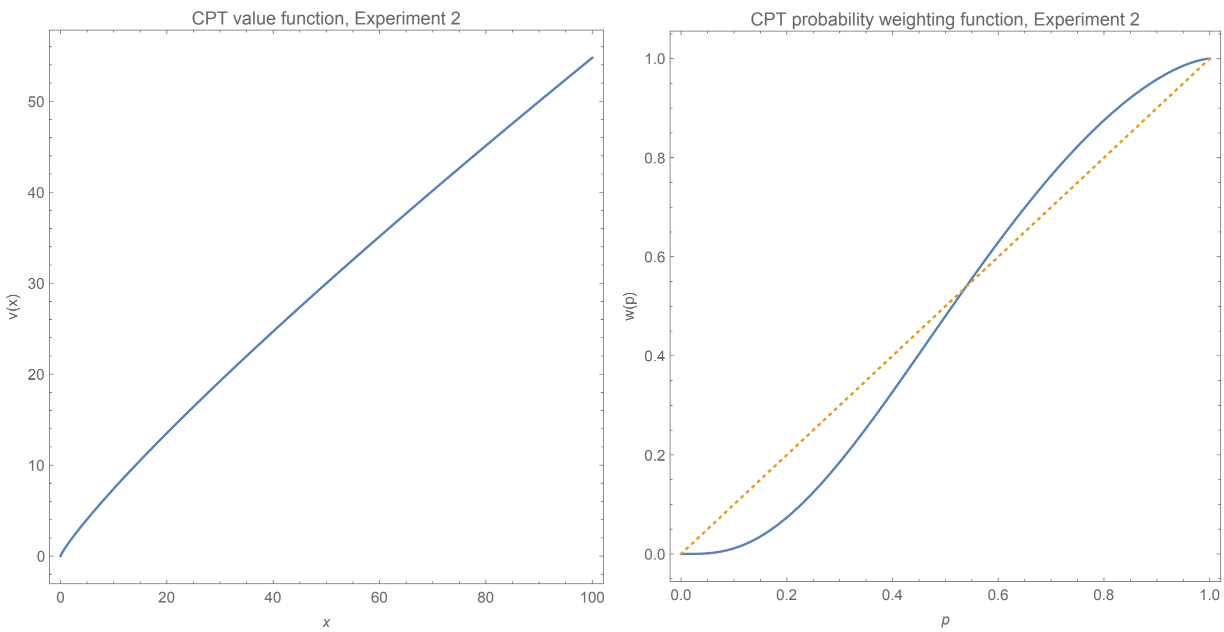

Figure 5.4: CPT value function (left) and probability weighting function (right) estimated using data from Experiment 2.

6. Discussion

The experimental results presented in this paper create at least two problems for the theory of decision-making under risk. First, they indicate monotonicity violations. Lotteries (0, ⅓; 300, ⅔) and (0, ⅓; 300, ⅓; 300, ⅓) are equivalent from a normative perspective. However, in the experiment, the latter is valued more than the former. Not only that, but lotteries with three equally likely outcomes, e.g. (0, 285, 300), (0, 270, 300), and even (0, 225, 300), are valued more than the binary lottery (0, ⅓; 300, ⅔), despite being clearly inferior. The opposite is observed when the middle outcome value is low, i.e. three-outcome lotteries are valued less than a binary lottery, despite being clearly superior. This violation is also observed for four-outcome lotteries with a variety of payoff schedules and subjects from two countries.

Another problem concerns the rate at which the CEs of multi-outcome lotteries increase as the middle outcome value(s) increase. According to CPT, the lottery valuation is less sensitive to middle outcomes than to extreme ones. However, the estimated slopes of CE changes with respect to middle outcome(s) are greater than the probabilities of their occurrence. This evidence can only be explained by the CPT probability weighting function assuming an S-shape. This, however, is contradicted by the results for binary lotteries obtained in Experiment 1, and in other experiments reported in the literature.

The paper shows that both problems can be explained by incorporating the median outcome value into any modeling of decision-making under risk. A simple weighted EUT-Median model is more accurate than CPT in both experiments. Importantly, it is consistent: a single weight value assigned to the median can satisfactorily explain the data for two, three-, and four outcome lotteries (contrary to CPT, which requires reversed shapes for binary and multi-outcome lotteries). Moreover this model predicts "overweighting" of small probabilities, and "underweighting" of large ones in the case of binary lotteries. According to the model, people tend to shift their CE valuations towards the median outcome value (the mid-point in the case of binary lotteries). This results in smaller CE values increasing when the probabilities of winning are small and large CE values decreasing when the probabilities of winning are large. Note that this shift occurs on the CE value level, and not on the probability level; probabilities are not weighted in any sense.

In the case of multi-outcome lotteries, the tendency to shift CE valuations towards the median outcome value works in the opposite direction. Obviously, the median value must be lower than the mean for right-skewed distributions (i.e. when most values are located below the mid-point). In this case, this tendency results in a lower CE value. Per contra, the median value is obviously greater than the mean for left-skewed distributions (i.e. when most values are located above the mid-point). In this case, the tendency results in a greater CE value. Again, this shift occurs on the CE value level; probabilities remain untouched. However, to "translate" this explanation into CPT probability terms: small probabilities appear to be "underweighted", and large probabilities appear to be "overweighted" in the case of multi-outcome lotteries, in contradistinction to the pattern observed for binary lotteries.

Note further that when an outcome distribution changes from right- to left-skewed (i.e. when most outcomes are moved from below to above the mid-point), the median value changes faster than the mean. This explains why estimated slopes in linear models have values greater than the probabilities of outcome occurrence.

The monotonicity violations can be explained as follows. The median outcome in a binary lottery is the mid-point between the lower and upper outcomes. Splitting the upper outcome into two parts results in a sudden change of the median value from the mid-point to the upper outcome. This leads to a sudden increase in lottery valuation and monotonicity violations. Per contra, splitting the lower outcome into two parts results in a sudden change of the median value from the mid-point to the lower outcome. This leads to a sudden decrease in the lottery valuation and monotonicity violations.

The explanations presented are consistent and satisfactorily explain the data collected in both experiments. The psychology of the phenomenon is clear and intuitive. On the other hand, these results raise the question as to whether probability weighting is the correct phenomenon to explain decisions made under conditions of risk, when, according to the data collected, the shapes of the probability weighting function have to be reversed for binary and multi-outcome lotteries.

7. References

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8. Appendixes

Appendix 1: Instruction used in Experiment 1 (a translation from Polish).

You have a choice to either invest in a risky venture or earn a sure sum of money.

The risky venture may have 2, 3, or 4 scenarios. These may be:

a) a pessimistic and an optimistic scenario (when two scenarios are present);

b) a pessimistic, a neutral, and an optimistic scenario (when three scenarios are present);

c) a very pessimistic, a moderately pessimistic, a moderately optimistic, and a very optimistic scenario (when four scenarios are present).

The scenarios, from the most pessimistic to the most optimistic, are presented in subsequent columns in a table.

The probabilities of occurrence for each scenario are given in the upper row of the table.

The amounts you can earn in each scenario are given in the lower row of the table.

If you invest in a risky venture, your payoff will depend on which scenario occurs. The risk lies in the fact that you have no influence over this.

On the other hand, you can avoid taking part in a risky venture, and earn a sure sum of money (i.e. with 100% certainty) whatever the scenario (e.g. you can put your money into a savings account).

Task:

State the sure sum of money that would make you indifferent between accepting it and taking part in a risky venture, i.e. so that it would not matter to you whether you received this sure sum or took part in the venture.

Example 1:

Problem xx

33,3%	33,3%	33,3%		100%	
0 zł	100 zł	300 zł	=		zł

According to the table, you will earn 0 zł in the pessimistic scenario, 100 zł in the neutral scenario, and 300 zł in the optimistic scenario. Each scenario has a 33.3% chance of occurrence.

a) Think of the sure sum of money that would make you indifferent between receiving it and taking part in a risky venture. Write this value in the field below the figure 100% on the right side (100% means that you would receive this amount for sure).

b) If you feel that you would prefer to receive this sum than take part in a risky venture, then the value you have written is too high.

c) If you feel that you would prefer to take part in a risky venture than receive the sum, then the value you have written is too low.

d) Repeat steps a), b), c) until you are indifferent as to whether you take part in a risky venture or receive the sure sum of money.

Further comments:

Carefully consider the amounts given in the problems, and remember that you stand to gain real money. In fact, some of you will be selected to take part in a real risky venture after the experiment is finished.

Note that payoffs vary across problems.

Try to state the sure sum of money as precisely as possible – at least to within 5-20 zł. Avoid giving rounded amounts. The more precise your answers, the greater their academic worth.

Do not try to be “mathematically correct”. Obviously, you are not prohibited from counting. It might even be advisable that you do so. Keep in mind, however, that this is a psychological, and not a mathematical, test.

Before you complete the experiment, try one more example.

Example 2:

…………….

If you understand the instructions, start the test by clicking “Next”.

If you are not sure about anything, read the instructions again.

If you do not wish to complete the test, press “Return”.

Appendix 2: Instruction used in Experiment 2.

Instructions for Cash Values of Gambles:

In this task, you are asked to judge the cash equivalent values of gambles. Each gamble can be thought of as a container holding several tickets. Each ticket has a prize value printed on it. You get to reach in the container and draw out one ticket blindly and at random, and the value printed on the ticket is your prize.

For example, consider the following case of a container holding exactly 4 tickets:

($30, $40, $50, $95) Cash Value =

Each ticket is equally likely, so you might win $30, $40, $50, or $95. Would you like to get one of these prizes? Yes, you would. How much is the opportunity to reach in and draw out a ticket worth? We are not asking you to judge what you would pay for the opportunity to draw a ticket and win a prize, but it might be helpful for you to think of that amount as a starting place. Instead, you are asked to state an amount of money such that you would like the (sure) cash and the gamble equally well. To help you make a judgment, you might write down an amount in the box provided below, and then ask yourself, which would I rather have? Would I prefer the cash for sure, or would I rather take a chance and try the gamble? If you prefer the cash, then the amount you wrote down is too big. If you prefer the gamble, then the amount you wrote down was too small. If you like each option equally well, then your answer is just right. You should feel equally attracted to the money and the gamble.

Now, if you wrote something less than $30, then you would rather take the gamble, because the LEAST you could win with the gamble is $30, and you might win as much as $95. But the most you can win in any of these choices is $95, so your answer will be less than that. At first, the task may be a bit difficult but you will soon be able to judge the cash values of the gambles. The first few trials are for practice. If you are not sure what to do, re-read the instructions, and if still unsure, raise your hand and ask for help.

You will notice that the number of tickets in the urn varies from two to ten. When there is only one ticket, or if all tickets have the same prize, that is a sure thing to win that amount, so its cash value is the same as the value printed on the ticket. For example, if it says $30, then you would be indifferent whether to take the money ($30) or to reach in the container and draw $30. When there are two tickets, each ticket has a fifty-fifty chance. For example, ($30, $100) is a fifty-fifty chance to win either $30 or $100. The worst you could do with that gamble is $30 and the best you could do is $100.

($5, $95) Cash Value =

Type a number in the box above and then ask yourself which you would rather have. The amount you typed or the gamble. If they are not equal, you should adjust the amount in the box so that they are equally good, in your opinion. Remember, the worst you could do with the gamble is $5, so your judgment should be greater than $5, and the most you could win is $95, so your judgment should be less than $95.

Please re-read the instructions to make sure you understand the task. When you understand the instructions, write in the amount of cash equal to each gamble below:

W1. ($5, $95)

Appendix 3: Aggregated CE values obtained in Experiment 1

Binary lotteries (p - probability of obtaining x_{max}, $x_{min} = 0$)

p	0.01	0.05	0.1	0.25	0.5	0.75	0.9	0.95	0.99
CE: x_{max}=300	15.4	32.9	39.8	78.4	152.1	217.3	258.9	275.0	292.7
CE: x_{max}=900	24.0	69.3	97.3	220.6	442.6	669.7	781.0	829.9	880.6

Three-outcome lotteries ($x_{min} = 0$, $x_{max} = 300$)

x	0	15	30	75	150	225	270	285	300
CE: x_{max}=300	93.6	93.4	104.6	119.8	146.9	193.9	207.6	219.4	214.7

Three-outcome lotteries ($x_{min} = 0$, $x_{max} = 900$)

x	0	45	90	225	450	675	810	855	900
CE: x_{max}=900	290.2	246.6	286.3	357.1	461.3	584.3	623.6	627.0	646.5

Four-outcome lotteries ($x_{min} = 0$, $x_{max} = 300$)

CE:x2|x3	0	30	150	200	270	300
0	76.4	79.5	130.		157.7	159.3
30		87.5	124.7		163.2	169.
100				170.3		
150			155.7		195.1	205.8
270					244.5	248.4
300						250.6

Four-outcome lotteries ($x_{min} = 0$, $x_{max} = 900$)

CE:x2|x3	0	90	450	600	810	900
0	202.7	205.1	376.2		480.9	436.6
90		227.6	338.		480.1	511.
300				452.		
450			482.2		586.4	572.8
810					711.7	712.3
900						723.1

Appendix 4: Aggregated CE values obtained in Experiment 2

Three-outcome lotteries (x_{min} = 5, x_{max} = 95)

x	10	20	30	40	50	60	70	80	90
CE1	20.7	26.5	32.6	39.8	47.5	53.7	58.8	65.6	64.8
CE2	21.4	27.2	34.	39.7	48.3	55.2	59.2	69.1	70.8

Four-outcome lotteries (x_{min} = 5, x_{max} = 95)

CE:x2|x3	10	20	30	40	50	55	60	70	80	90
10	18.	22.5	27.5	34.7	36.4		41.	40.1	38.7	45.
20	22.9	25.	29.3	33.	36.8		41.	42.3	41.2	41.8
30	27.4	28.4	32.1	36.3	38.6		41.6	44.7	44.3	45.1
40	33.1	33.7	38.6	40.5	44.5		46.7	50.2	47.4	48.5
45						48.3				
50	37.4	38.4	41.7	42.8	49.2		51.2	54.5	53.2	54.2
60	42.8	44.	42.4	48.4	52.5		58.4	55.4	59.6	62.2
70	42.1	42.5	44.4	49.3	52.7		59.3	66.9	63.7	67.
80	44.8	40.	43.3	49.5	55.5		58.7	64.1	72.7	70.8
90	40.2	46.2	44.8	51.	55.3		60.6	66.5	73.8	80.